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Generalized Differentiability of Continuous Functions

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Abstract: Many physical phenomena give rise to mathematical models in terms of fractal, non-differentiable functions. The paper introduces a broad generalization of the derivative in terms of the maximal modulus of continuity of the primitive function. These derivatives are called indicial derivatives. As an application, the indicial derivatives are used to characterize the nowhere monotonous functions. Furthermore, the non-differentiability set of such derivatives is proven to be of measure zero. As a second application, the indicial derivative is used in the proof of the Lebesgue differentiation theorem. Finally, the connection with the fractional velocities is demonstrated.

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1. Introduction

From a certain perspective, a derivative can be interpreted as an idealization of the linear approximation. Moreover, the derivatives are indispensable tools for physical modelling. In classical physics, the second law of Newton tacitly assumes that the velocity is a differentiable function of time. This ensures mathematical modelling in terms of differential equations, and hence (almost everywhere) differentiable functions. Ampere even tried to prove that all functions are almost everywhere differentiable [1]. Now we know that this attempt was doomed to fail. Various non-differentiable functions were constructed in the 19th century and regarded with a mixture of wonder and horror (historical survey in [2]). For example, Charles Hermite wrote "I turn with terror and horror from this lamentable scourge of functions with no derivatives", while Henri Poincaré denounced these functions as "an outrage against common sense". The interest in fractal and non-differentiable functions was rekindled with the works of Mandelbrot in fractals [3]. Moreover, many physical phenomena give rise to models in terms of fractal, non-differentiable functions. Such an opposition leads to the question whether the ordinary (i.e., Cauchy) derivatives can be generalized locally in a way that can describe such irregular objects. This is certainly achievable for a class of singular functions, such as the De Rham's function, which can be characterized locally in terms of *fractional velocity* [4,5]. Purely mathematically, the derivatives can be generalized in several ways. Derivatives can be defined in the usual way as limits of difference quotients on the accumulation sets of points [6], (Chaper 3, p. 105). This approach can be applied also to functions defined on perfect fractal sets as demonstrated in [7–9].

Having physical applications in mind, it is logical to restrict the functions of study to continuous ones. Such functions will be allowed to vary everywhere on the real line. So this condition makes the setting very different from the approach of Pravate, Gangal and Yang.

A very broad generalization of derivatives can be achieved if the linear, Lipschitz, growth condition is replaced by a non-linear one. From this perspective, the growth condition can be taken as the modulus of continuity. Such notion is by necessity limited. However, it provides a framework which allows one to identify if a certain property of derivatives carries over to more particular constructs, for example in the setting of fractional velocities [10], or it is special for the linear growth condition.

The paper is organized as follows. Section 2 introduces some general definitions and notations. Section 3 introduces the oscillation of functions. Section 4 introduces indicial derivatives. Section 5 gives the construction of a singular function used to clarify the Lebesgue differentiation theorem. Section 6 characterizes the continuity sets of derivatives. Section 7 characterizes the fractional velocities in terms of the indicator derivatives.

2. General Definitions and Conventions

The term variable denotes an indefinite number taken from a predefined set, usually the real numbers. Sets are denoted by capital letters, while variables taking values in sets are denoted by lowercase. The term function denotes a mapping from one number to another and the action of the function is denoted as $f(x) = y$. Implicitly the mapping acts on the real numbers: $f : R \mapsto R$. If a statement of a function f fulfils a certain predicate with argument A (i.e., $Pred[A]$) the following short-hand notation will be used $f \cong Pred[A]$.

The co-domain of the function $f : X \mapsto Y$ is denoted as $f[X] = Y$. The term operator denotes the mapping from a functional expression to functional expression. The term functional denotes the mapping from a functional expression to a number. Square brackets are used for the arguments of operators and functionals, while round brackets are used for the arguments of functions. The term Cauchy sequence will always be interpreted as a null sequence.

Everywhere, ϵ will be considered as a small positive variable. The asymptotic small $\mathcal{O}$ notation will be used as introduced in [5].

Definition 1. *Define the parametrized difference operators acting on the function $f(x)$ as*

$$\Delta_\epsilon^+ [f] (x) := f(x + \epsilon) - f(x),$$
$$\Delta_\epsilon^- [f] (x) := f(x) - f(x - \epsilon)$$

for the variable $\epsilon > 0$. The two operators are referred to as forward difference and backward difference operators, respectively.

Definition 2 (Anonymous function notation). *The notation for the pair $\mu :: \epsilon$ will be interpreted as the implication that if Left-Hand Side (LHS) is fixed then the Right-Hand Side (RHS) is fixed by the value chosen on the left, i.e., as an anonymous functional dependency $\epsilon = \epsilon(\mu)$.*

3. The Discontinuity Sets of Functions

In order to state the main result of the section we introduce the auxiliary concept of function oscillation.

Definition 3 (Oscillation). *Define the oscillation of the function f in the interval $J = [a, b]$ as*

$$\operatorname*{osc}_J f := \sup_J f - \inf_J f$$

Definition 4 (Directed Oscillation). *Define the directed oscillations as (i) the forward oscillation:*

$$\mathrm{osc}_{\epsilon}^{+}[f](x) := \sup_{[x,x+\epsilon]} f - \inf_{[x,x+\epsilon]} f, \quad I = [x, x+\epsilon]$$

and the backward oscillation:

$$\mathrm{osc}_{\epsilon}^{-}[f](x) := \sup_{[x-\epsilon,x]} f - \inf_{[x-\epsilon,x]} f, \quad I = [x-\epsilon, x]$$

Finally, define the limits, if such exist as finite numbers, as

$$\mathrm{osc}^{+}[f](x) := \lim_{|I|\to 0} \left(\sup_{I} f - \inf_{I} f \right), \quad I = [x, x+\epsilon]$$

$$\mathrm{osc}^{-}[f](x) := \lim_{|I|\to 0} \left(\sup_{I} f - \inf_{I} f \right), \quad I = [x-\epsilon, x]$$

where $|\cdot|$ denotes the length of the interval, according to previously introduced notation [5,11].

The lemma was stated in [11]:

Lemma 1 (Oscillation lemma). *Consider the function $f : X \mapsto Y \subseteq \mathbb{R}$. Suppose that $I_{+} = [x, x+\epsilon] \subseteq X$, $I_{-} = [x-\epsilon, x] \subseteq X$, respectively.*
If $\mathrm{osc}^{+}[f](x) = 0$ then f is right-continuous at x. Conversely, if f is right-continuous at x then $\mathrm{osc}^{+}[f](x) = 0$. If $\mathrm{osc}^{-}[f](x) = 0$ then f is left-continuous at x. Conversely, if f is left-continuous at x then $\mathrm{osc}^{-}[f](x) = 0$. That is,

$$\lim_{\epsilon\to 0} \mathrm{osc}_{\epsilon}^{\pm}[f](x) = 0 \iff \lim_{\epsilon\to 0} f(x \pm \epsilon) = f(x)$$

Then the negation of the statement is also true.

Corollary 1. *The following two statements are equivalent*

$$\lim_{\epsilon\to 0} \mathrm{osc}_{\epsilon}^{\pm}[f](x) > 0 \iff \lim_{\epsilon\to 0} f(x \pm \epsilon) \neq f(x)$$

Theorem 1 (Disconnected discontinuity set). *Consider a bounded function F defined on the compact interval I. Then its set of discontinuity $\Delta[F, I]$ is totally disconnected in I.*

Proof. Consider a decreasing collection of closed nested intervals from a partition of the interval $I_1 = [x, x+h]$ ($h > 0$, not necessarily small); that is $\{I_k = [x, x+a_k]\}_{k=1}^{n}$, $I_{k+1} \subset I_k \subset \ldots \subset I_1$.
Since the case when the function is locally constant is trivial we consider only two cases: increasing and decreasing. Let $E_n = I_n \cap I_{n-1}$ and define

$$\Delta_n := \sup_{I_n} F - \inf_{I_n} F$$

$$\Delta_{n-1} := \sup_{I_{n-1}} F - \inf_{I_{n-1}} F$$

$$\Delta_E := \sup_{E_n} F - \inf_{E_n} F$$

Increasing case: Suppose that F is increasing in I_{n-1} then

$$\Delta_{n-1} - \Delta_n = \sup_{I_{n-1}} F - \inf_{I_{n-1}} F - \sup_{I_n} F + \inf_{I_n} F = \sup_{E_n} F - \inf_{I_{n-1}} F - \inf_{E_n} F + \inf_{I_{n-1}} F = \Delta_E = \underset{E_n}{osc}\, f$$

Decreasing case: Suppose that F is decreasing in I_{n-1} then

$$\Delta_{n-1} - \Delta_n = \sup_{I_n} F - \inf_{E_n} F - \sup_{I_n} F + \sup_{E_n} F = \Delta_E = \underset{E_n}{osc}\, f$$

On the other hand, $\underset{I_n}{osc}\, f = \Delta_n$ and $\underset{I_{n-1}}{osc}\, f = \Delta_{n-1}$. If F is continuous on the opening of I_{n-1}, in limit

$$\lim_{n\to\infty} \underset{I_n}{osc}\, f - \underset{I_{n-1}}{osc}\, f = 0 \Rightarrow \lim_{|E_n|\to 0} \underset{E_n}{osc}\, f = 0$$

and $\Delta[F, I] \setminus \{x\} = \varnothing$. On the other hand, F can be discontinuous on the boundary points $\{x, x + a_{n-1}\}$, which are disconnected.

Suppose then that both endpoints are points of discontinuity. Hence, $\{x, x + a_{n-1}\} \subseteq \Delta[F, I]$. If F is discontinuous on the opening of I_{n-1}, we take I_n and proceed in the same way. Therefore, $\Delta[F, I] \neq \varnothing$ is a union of totally disconnected sets and hence it is totally disconnected. $\square$

The disconnected discontinuity theorem is used as the main tool characterizing the continuity sets of derivatives.

4. Indicial ω Derivatives

Definition 5 (Modulus of continuity). *A modulus of continuity $g_x : \mathbb{R} \mapsto \mathbb{R}$ is a*

(1)   non-decreasing continuous function, such that
(2) $g_x(0) = 0$ and
(3) $|\Delta_\epsilon^{\pm}[f](x)| \leq K\, g_x(\epsilon)$ holds in the interval $I = [x, x \pm \epsilon]$ for some constant $K > 0$.

I is assumed to be in the domain of the function f. In addition, a regular modulus will be such that $g_x(1) = 1$.

Definition 6. *Consider a bounded function f defined on a compact interval I. Define the right (resp. left) point oscillation functions as*

$$\omega_x^{\pm}(\epsilon) := \begin{cases} \sup_I |f(x+\epsilon) - f(x)|, & I = [x, x+\epsilon] \\ \sup_I |f(x) - f(x-\epsilon)|, & I = [x-\epsilon, x] \end{cases}$$

This section introduces the indicator or ω-derivatives. I propose this name based on the properties of their image sets. These derivative behave in a way like the signed indicator function of a set.

Definition 7. *Define the right and left modulus-variation operators as*

$$v_\omega^{\epsilon+}[f](x) := \frac{\Delta_\epsilon^+[f](x)}{\omega_x^+(\epsilon)} \tag{1}$$

$$v_\omega^{\epsilon-}[f](x) := \frac{\Delta_\epsilon^-[f](x)}{\omega_x^-(\epsilon)} \tag{2}$$

for a positive ϵ.

Definition 8. *For a function f define superior and inferior, and respectively forward and backward, Dini-type ω derivatives, as the limit numbers L*

$$\left| \sup_{\epsilon} v_{\omega}^{\epsilon\pm}\left[f\right](x) - L \right| < \mu \Longrightarrow \bar{\mathcal{D}}_{\omega}^{\pm} f(x) = L$$

$$\left| \inf_{\epsilon} v_{\omega}^{\epsilon\pm}\left[f\right](x) - L \right| < \mu \Longrightarrow \mathcal{D}_{\omega}^{\pm} f(x) = L$$

for all $\epsilon :: \mu, \epsilon > 0$.

Remark 1. *These derivative functions obviously generalize the concept of Dini derivatives:*

$$\bar{\mathcal{D}}^{\pm} f(x) = \limsup_{\epsilon \to 0} \frac{\Delta_{\epsilon}^{\pm}\left[f\right](x)}{\epsilon}$$

$$\mathcal{D}^{\pm} f(x) = \liminf_{\epsilon \to 0} \frac{\Delta_{\epsilon}^{\pm}\left[f\right](x)}{\epsilon}$$

for the function f. It can be seen also that all four such ω-derivatives are of Baire I class (Definition A3).

Equipped with the above definition we can state the first existence result:

Theorem 2 (Bounded ω-derivatives). *For a continuous function the four derivative functions exist as real numbers. Moreover, if f is non-decreasing about x^{+}*

$$\bar{\mathcal{D}}_{\omega}^{\pm} f(x) = 1$$

$$0 \leq \mathcal{D}_{\omega}^{\pm} f(x) \leq 1$$

while if f is non-increasing about x^{-}

$$\mathcal{D}_{\omega}^{\pm} f(x) = -1$$

$$0 \geq \bar{\mathcal{D}}_{\omega}^{\pm} f(x) \geq -1$$

Proof. Let $I = [x, x \pm \epsilon]$ be given and x is fixed but we can vary ϵ. Consider the auxiliary function

$$v_{\omega}^{\epsilon\pm}\left[f\right](x) := \frac{\Delta_{\epsilon}^{\pm}\left[f\right](x)}{\omega_{x}^{\pm}(\epsilon)} \tag{3}$$

The supremum definitions are restatements of the Least Upper Bound (LUB) property for $v_{\omega}^{\epsilon\pm}\left[f\right](x)$ in terms of the variable ϵ, while the infimum derivatives are restatements with the Greatest Lower Bound (GLB) property of the reals again for the same variable. Therefore, all four numbers exist for a given argument x and, therefore, under the above hypothesis. Moreover, since $\left|\Delta_{\epsilon}^{\pm}\left[f\right](x)\right| \leq \omega_{x}^{\pm}(\epsilon)$ then for an non-decreasing function $\left|\bar{\mathcal{D}}_{\omega}^{\pm} f(x)\right| \leq 1$. Therefore, by the supremum property $\sup_{\epsilon} \frac{\Delta_{\epsilon}^{\pm}[f](x)}{\omega_{x}^{\pm}(\epsilon)} = 1$. For a decreasing function f it is sufficient to consider $-f$ and apply the same arguments. $\square$

We can give generalized definition of local differentiability (called ω-differentiability) as follows:

Definition 9. *A function f is ω-differentiable at x if at least one of the two limits exist*

$$\left| \frac{\Delta_{\epsilon}^{\pm}\left[f\right](x)}{\omega_{x}^{\pm}(\epsilon)} - L \right| < \mu \Longrightarrow \mathcal{D}_{\omega}^{\pm} f(x) = L$$

where the conventions for L, μ and ϵ are as in Definition 8.

Note that the definition only supposes that the one-sided limits of the increments—$\mathcal{D}_\omega^+ f(x)$ (respectively $\mathcal{D}_\omega^- f(x)$) exist as real numbers. That is,

$$\mathcal{D}_\omega^+ f(x) \neq \mathcal{D}_\omega^- f(x)$$

is admissible. This is the minimal statement that can be given for the limit of an increment of a function. Nevertheless, based on two strong properties—monotonicity and continuity—it can be claimed that

Theorem 3 (Monotone ω-differentiation). *If a function f is monotone and continuous in a compact interval I then it is continuously ω-differentiable everywhere in the opening I°. If f is increasing in $[x, x + \epsilon] = I$ then $\bar{\mathcal{D}}_\omega^+ f(x) = \mathcal{D}_\omega^+ f(x) = 1$. If f is decreasing in $[x, x + \epsilon] = I$ then $\bar{\mathcal{D}}_\omega^+ f(x) = \mathcal{D}_\omega^+ f(x) = -1$. If f is increasing in $[x - \epsilon, x] = I$ then $\bar{\mathcal{D}}_\omega^- f(x) = \mathcal{D}_\omega^- f(x) = 1$. If f is decreasing in $[x - \epsilon, x] = I$ then $\bar{\mathcal{D}}_\omega^- f(x) = \mathcal{D}_\omega^- f(x) = -1$.*

Proof. The continuity follows directly from Theorem 3, while the restriction comes from the fact that at the boundary only one of the increments can be defined without further hypothesis for the values of f outside of I. The evaluation of the derivatives comes as follows. Fix x and consider $I = [x, \pm\epsilon]$. The proof follows from the fact that in both cases $|\Delta_\epsilon^\pm [f](x)| = \omega_x^\pm(\epsilon)$. □

Definition 10. *Functions of the bounded variation (see Definition A1) on a compact interval I will be denoted by $BV[I]$. If further the function is continuous on I it will be denoted by $BVC[I]$.*

Theorem 4 (BVC ω-differentiation). *If the function f is $BVC[I]$ in a closed interval I then it is ω-differentiable everywhere in the opening I°.*

Proof. The proof follows from the Jordan theorem, since a $BV[I]$ function can be decomposed into a difference of two non-decreasing functions. On the other hand, without further hypotheses we can not claim anything about the eventual equality of $\mathcal{D}_\omega^+ f(x)$ and $\mathcal{D}_\omega^- f(x)$ since $J_x = [x - \epsilon, x] \cap [x, x + \epsilon] = \{x\}$ so that we can form only the trivial map $x \mapsto \{x\}$, which without further restrictions of the domain of x (i.e., by means of some topological obstructions) is uncountable. □

We can further utilize the concept of oscillation to give a concise general differentiability condition as

$$\lim_{\epsilon \to 0} \mathrm{osc}_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)} = 0 \tag{4}$$

Theorem 5 (Characterization of ω-derivative). *The following implications hold*

$$\bar{\mathcal{D}}_\omega^\pm f(x) = \mathcal{D}_\omega^\pm f(x) = \mathcal{D}_\omega^\pm f(x) \implies f \cong \mathcal{C}[x^\pm]$$

$$\lim_{\epsilon \to 0} \mathrm{osc}_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)} = 0 \iff \bar{\mathcal{D}}_\omega^\pm f(x) = \mathcal{D}_\omega^\pm f(x) = \mathcal{D}_\omega^\pm f(x)$$

so that if Equation (4) holds at x then f is ω-differentiable (and hence continuous) at x.

Proof. Continuity implication Consider the inequality

$$\bar{\mathcal{D}}_\omega^\pm f(x) = \mathcal{D}_\omega^\pm f(x) \implies \left| \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)} - L \right| \leq \mu/2, \ \epsilon :: \mu$$

so that

$$L - \mu/2 \leq \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} \leq L + \mu/2 \Longrightarrow$$

$$\sup_\epsilon \Delta_\epsilon^\pm [f](x) \leq (L + \mu/2)\, \omega_x^\pm (\epsilon)$$

$$(L - \mu/2)\, \omega_x^\pm (\epsilon) \leq \inf_\epsilon \Delta_\epsilon^\pm [f](x) \Longrightarrow$$

$$\left| \frac{\sup_\epsilon \Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} - L \right| \leq \mu/2$$

$$\left| \frac{\inf_\epsilon \Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} - L \right| \leq \mu/2$$

Let $\sup_\epsilon \Delta_\epsilon^\pm [f](x) = M$ and $\inf_\epsilon \Delta_\epsilon^\pm [f](x) = m$. Then by the triangle inequality

$$\frac{\sup_\epsilon \Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} - \frac{\inf_\epsilon \Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} = \frac{M - m}{\omega_x^\pm (\epsilon)} \leq \mu$$

$$M - m \leq \mu\, \omega_x^\pm (\epsilon)$$

Therefore, in limit $M - m \leq 0$, hence $M = m$ and f is continuous. This sequence of operations reminds the fact that real numbers are constructed by a limiting process.

Forward statement: Suppose that $\bar{\mathcal{D}}_\omega^\pm f(x) = L_1$ and $\mathcal{D}_\omega^\pm f(x) = L_2$ Then by LUB

$$\left| \sup_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} - L_1 \right| \leq \mu/2$$

$$\left| \inf_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} - L_2 \right| \leq \mu/2$$

so that

$$\left| \sup_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} - L_1 \right| + \left| \inf_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} - L_2 \right| \leq \mu$$

Then by the triangle inequality

$$|L_1 - L_2| \leq \left| \underbrace{\sup_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} - \inf_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)}}_{\text{osc}_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)}} + L_1 - L_2 \right| \leq \mu$$

Then in limit, by Lemma 1

$$|L_1 - L_2| \leq 0 \Longrightarrow L_1 = L_2$$

Further, starting from

$$\inf_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} \leq \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} \leq \sup_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} \Longrightarrow$$

$$0 \leq \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} - \inf_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} \leq \sup_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} - \inf_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)} = \text{osc}_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm (\epsilon)}$$

Therefore,

$$\left| \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)} - \inf_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)} \right| \leq \mu \implies \left| \sup_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)} - \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)} \right| \leq \mu$$

Therefore, all three limits coincide.

Converse statement: Suppose that

$$\bar{\mathcal{D}}_\omega^\pm f(x) = \mathcal{D}_\omega^\pm f(x) = L > 0$$

By hypothesis

$$\left| \underbrace{L - \inf_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)}}_{A_\epsilon} \right| \leq \mu/2$$

$$\left| \underbrace{\sup_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)} - L}_{B_\epsilon} \right| \leq \mu/2$$

$$\underbrace{\sup_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)} - \inf_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)}}_{\mathrm{osc}_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)}} \leq |A_\epsilon| + |B_\epsilon| \leq \mu$$

Therefore, in limit

$$0 \leq \lim_{\epsilon \to 0} \mathrm{osc}_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)} \leq 0$$

so that $\lim_{\epsilon \to 0} \mathrm{osc}_\epsilon \frac{\Delta_\epsilon^\pm [f](x)}{\omega_x^\pm(\epsilon)} = 0$. □

Corollary 2 (Range of $\mathcal{D}_\omega^\pm$)**.** *The range of $\mathcal{D}_\omega^\pm$ is given by the discrete set*

$$\mathcal{D}_\omega^\pm f(x) \subseteq \{-1, 0, +1\}$$

Proof. Let $I = [x, x + \epsilon]$ be given. If f is constant in I trivially $\mathcal{D}_\omega^\pm f(x) = 0$. If f is increasing in I then $\mathcal{D}_\omega^\pm f(x) = 1$ and by duality if f is decreasing in I then $\mathcal{D}_\omega^\pm f(x) = -1$. □

The ω non-differentiability set of a continuous function can be characterized by the following theorem.

Theorem 6 (ω non-differentiability set)**.** *Consider the function $f \cong \mathcal{C}[I]$ on the compact interval I. Then the sets*

$$\Delta_\omega^+[I] := \{x : \bar{\mathcal{D}}_\omega^+ f(x) > \mathcal{D}_\omega^+ f(x)\} \cap I, \quad \Delta_\omega^-[I] := \{x : \bar{\mathcal{D}}_\omega^- f(x) > \mathcal{D}_\omega^- f(x)\} \cap I$$

are null sets. That is, for a continuous function the ω non-differentiability set is null (see Definition A2).

Proof. Consider the case wherever the right ω-derivative does not exist. That is, the defining quotient oscillates without a limit. Then for $0 < u, v \leq \delta$

$$\left| \frac{\Delta_u^+ [f] (x)}{\omega_x^+ (u)} - \frac{\Delta_v^+ [f] (x)}{\omega_x^+ (v)} \right| > \mu \tag{5}$$

for some $\mu > 0$. We can consider a variable $\xi \in [x, x + u] \cap [x, x + v] = [x, x + \min(u, v)] = J$. There is a rational $r \in \mathbb{Q} \cap J$. Associate $(r, J) \equiv J_r$ so that J_r can be counted by an enumeration of the rationals and index $\delta :: r$. Therefore, the set

$$\Delta_\omega := \bigcup_{k=1}^{\infty} \{ z : \text{ineq. (5) is true}, z \in J_k \}$$

is countable $\forall \delta > 0$. Since Δ_ω is totally disconnected by Theorem 1 we can choose $\delta_k = \delta / 2^k$ and $J_k \subset J_r$. Therefore,

$$|\Delta_\omega| = \sum_{k=1}^{\infty} |J_k| \leq \sum_{k=1}^{\infty} \frac{\delta}{2^k} = \delta$$

so that Δ_ω is a null set by Definition A2. The left case holds by duality. $\quad\square$

5. Fat Cantor Sets and Related Quasi-Singular Functions

There are sets that are totally disconnected, uncountable and non-null. An example of such sets is the Smith–Volterra–Cantor set (i.e., the so-called Fat Cantor set), which is of measure $1/2$. The construction of the Smith–Volterra–Cantor set is given as follows: The set is constructed by iteratively removing certain intervals from the unit interval $I_0 = [0, 1]$. At each step k, the length that is removed $p_{k+1} = p_k/4$ from the middle of each of the remaining intervals. That is, starting from I_0 and $p_0 = 1/4$ on every step

$$I_k = [u, v] \longrightarrow I_{k+1}^l = [u, (u + v)/2 - p_k/2], \quad I_{k+1}^r = [(u + v)/2 + p_k/2, v]$$
$$p_k \longrightarrow p_{k+1} = p_k/4$$

That is $I_{k+1}^l \cup I_{k+1}^l = I_k \setminus ((u + v)/2 - p_k/2, \quad (u + v)/2 + p_k/2)$.
For example,

$$k = 1: \quad I_1 = \left[0, \tfrac{3}{8}\right], \ I_2 = \left[\tfrac{5}{8}, 1\right]$$

$$k = 2: \quad I_{21} = \left[0, \tfrac{5}{32}\right], \ I_{22} = \left[\tfrac{7}{32}, \tfrac{3}{8}\right], \ I_{23} = \left[\tfrac{5}{8}, \tfrac{25}{32}\right], \ I_{24} = \left[\tfrac{27}{32}, 1\right]$$

$$k = 3: \quad I_{31} = \left[0, \tfrac{9}{128}\right], \ I_{32} = \left[\tfrac{11}{128}, \tfrac{5}{32}\right], \ I_{33} = \left[\tfrac{7}{32}, \tfrac{37}{128}\right], \ I_{34} = \left[\tfrac{39}{128}, \tfrac{3}{8}\right],$$

$$I_{35} = \left[\tfrac{5}{8}, \tfrac{89}{128}\right], \ I_{36} = \left[\tfrac{91}{128}, \tfrac{25}{32}\right], \ I_{37} = \left[\tfrac{27}{32}, \tfrac{117}{128}\right], \ I_{38} = \left[\tfrac{119}{128}, 1\right]$$

During the process, disjoint intervals of total length

$$L = \sum_{k=0}^{\infty} \frac{1}{4} \cdot \frac{1}{2^k} = \frac{1}{2}$$

are removed so that the resulting set is of measure $1/2$. The Smith–Volterra–Cantor set is closed as it is an intersection of closed sets. Furthermore, at step n the length of each closed subinterval is $l_n = \frac{1}{2}(l_{n-1} - p_{n-1})$. Starting from $l_0 = 1$ one gets

$$l_n = \frac{1}{2}\left(\frac{1}{2^n} + \frac{1}{4^n}\right)$$

Therefore, by the Nested Interval Theorem the Smith–Volterra–Cantor (SVC) set is totally disconnected and contains no intervals. The set presented above can be used to construct a quasi-singular function, called Smith–Volterra–Cantor (SVC) function. The quasi-singularity will be defined as the property of the function to have a totally disconnected set of change χ. Formally,

Definition 11 (Quasi-singular function). *Denote the set $\{x : f'(x) = 0\}$ by $\mathcal{N}$. A continuous function f is called quasi-singular on a compact interval I if*

- *f is non-constant on I.*
- *the complement $\mathcal{N}^c \cap I$ is totally disconnected.*

The appearance of the function is jagged and its general outline resembles the Cantor's "Devil's staircase" (see Figure 1). The function was introduced in [12].

Definition 12 (SVC Function). *Define the SVC function as the continuous map between the SVC set defined above and the dyadic rationals $D \setminus \{1/2\}$ in the following construction. Let S_n be the sequence of the end-points of the interval in the n-th step in the construction of the SVC set. Let D_n be the sequence of dyadic rationals with denominator 2^n excluding $1/2$.*

Define the sequence of continuous piece-wise linear interpolating functions (see Figure 1) $F_n : [0,1] \mapsto [0,1]$, such that

$$F_n(S_n) = D_n$$

and pass to the limit $F_C(x) := \lim_{n \to \infty} F_n(x)$.
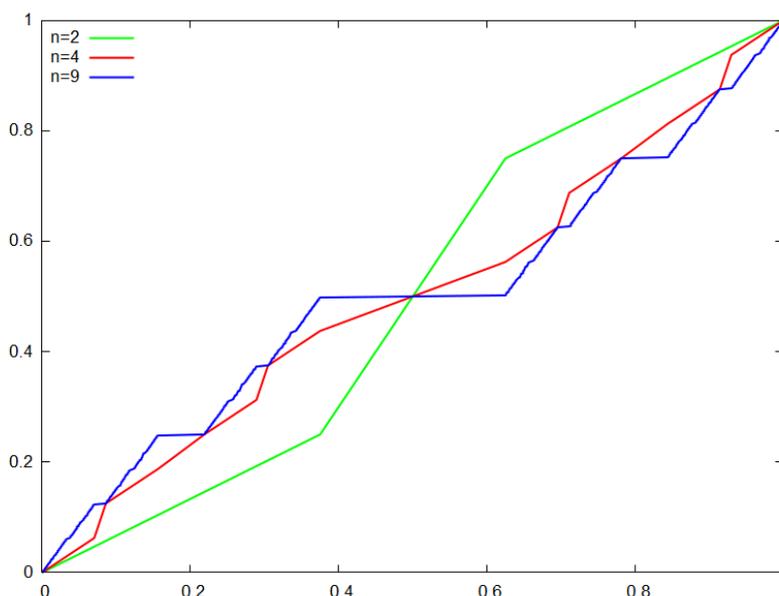
Figure 1. Approximations of the Smith–Volterra–Cantor (SVC) function.

In contrast to the Cantor-Lebesgue function, the SVC function is Lipschitz on its set of change. In the following argument we characterize the set of growth of the SVC function. Let $\Delta_{f,I} := \{x : f' \notin \mathcal{C}, x \in I\}$.

In the case of the SVC function $I = [0, 1]$. Then by construction, the set $C = \{x : F'_C(x) = 0\}$ has measure $1/2$. On the other hand, for $u \in S_n$ and $h_0 = 1$

$$h_k = \frac{1}{2}\left(h_{k-1} - \frac{1}{4^k}\right)$$

$$q_k = 2^k h_k = 2^{k-1} h_{k-1} - \frac{1}{2^{k+1}} = q_{k-1} - \frac{1}{2^{k+1}}$$

Then by induction

$$q_n = 1 - \frac{1}{2}\sum_{k=1}^{n}\frac{1}{2^k}$$

Therefore, in limit

$$q^* = 1 - \frac{1}{2}\left(\frac{1}{1 - 1/2} - 1\right) = \frac{1}{2}$$

Then for the difference quotient

$$\frac{\Delta_{h_n} F_n(u)}{h_n} = \frac{1}{q_n}$$

Therefore, in limit

$$\lim_{n\to\infty}\frac{\Delta_h F_n(u)}{h} = F'^-_C(u) = 2$$

while $F'^+_C(u) = 0$. Therefore, $|\Delta_{F_C, I}| = 1/2$.

Theorem 7. *Let S denote the SVC set and* $I = [0, 1]$*. Then* $F'^+_C(u) = 0$ *on* $u \in I \setminus S$ *and* $F'^-_C(u) = 2$ *on* $u \in S$*. Moreover,* $|F'^+_C(u) = 0| = |F'^-_C(u) = 0| = 1/2$*.*

Corollary 3. *The SVC function is quasi-singular.*

The presented theorem has an impact on the formulation of the Lebesgue monotone differentiation theorem. In fact, it requires that the hypothesis of Lebesgue monotone differentiation theorem require strict monotonicity of the function. This refinement is done in the following section.

6. Continuity Sets of Derivatives

This section re-states the classical result of the Lebesgue differentiation theorem. The poof is given using the machinery of ω-differentiation. In the following argument I reserve the term "strictly monotone function" to mean only a strictly increasing or strictly decreasing function in an interval.

Theorem 8 (Lebesgue monotone differentiation theorem)**.** *Suppose that f is strictly monotone and continuous in the compact interval I. Then f is continuously differentiable almost everywhere. The set*

$$\Delta_f[I] := \{x : f'_+(x) \neq f'_-(x)\} \cap I$$

is a null set (see Definition A2).

Proof. Let $I = (x - \epsilon, x + \epsilon)$. Since f is strictly monotone and continuous then its ω-derivative exists by Theorem 3. Therefore, let $\mathcal{D}_\omega f(x) = L > 0$. By Theorem 3 for $\epsilon :: \mu$

$$\underbrace{\left| L - \frac{\Delta_\epsilon^+ [f](x)}{\omega_x^+ (\epsilon)} \frac{\omega_x^+ (\epsilon)}{\epsilon} \right|}_{A_\epsilon} \leq \mu/2$$

$$\underbrace{\left| \frac{\Delta_\epsilon^- [f](x)}{\omega_x^- (\epsilon)} \frac{\omega_x^- (\epsilon)}{\epsilon} - L \right|}_{B_\epsilon} \leq \mu/2$$

$$\left| \frac{\Delta_\epsilon^+ [f](x)}{\omega_x^+ (\epsilon)} \frac{\omega_x^+ (\epsilon)}{\epsilon} - \frac{\Delta_\epsilon^- [f](x)}{\omega_x^- (\epsilon)} \frac{\omega_x^- (\epsilon)}{\epsilon} \right| \leq |A_\epsilon| + |B_\epsilon| \leq \mu$$

$$\left| \frac{\omega_x^+ (\epsilon)}{\epsilon} - \frac{\omega_x^- (\epsilon)}{\epsilon} \right| \leq \mu$$

Therefore, by monotonicity using the original notation

$$\left| \frac{\Delta_\epsilon^+ [f](x)}{\epsilon} - \frac{\Delta_\epsilon^- [f](x)}{\epsilon} \right| = \left| \frac{\Delta_\epsilon^2 [f](x)}{\epsilon} \right| \leq \mu$$

hence $f'_+ (x) = f'_- (x)$ and $\Delta_f[I] = \varnothing$. The argument above is not applicable to the boundary points $B = \{x - \epsilon, x + \epsilon\}$, however they are countable, since they can be covered by intervals with rational end-points, and hence the union of all B sets is a null set. $\square$

Recall the definitions of nowhere monotone functions:

Definition 13. *A function is nowhere monotone in the compact interval $I = [a, b]$, written NM[I], if given any $a < x < y < z < b$*

$$(f(y) - f(x))(f(z) - f(y)) \leq 0$$

so that NM[I] function is neither non-decreasing nor non-increasing on any sub-interval of I. A function, which is nowhere monotone at a point (NM[y]), is treated as above while y is held fixed.

From the Lebesgue monotone differentiation theorem it follows that a nowhere differentiable function on an open interval I is simultaneously nowhere monotone on I. Brown et al. establish that no continuous function of bounded variation BVC is MN[y] [13] (Theorem 12. Corr. 3). That is to say $NM[x]$ for $x \in I$ as above. Therefore, it is of interest to establish the following result.

Theorem 9 (NM continuous ω-differentiability). *Suppose that $f \cong \mathcal{C}[I]$ and $f \cong NM[I]$. Then*

$$\mathcal{D}_\omega f(x) \cong \mathcal{C}[I] \implies \mathcal{D}_\omega f(x) = 0$$

Proof. Suppose that $\mathcal{D}_\omega^\pm f(x)$ exists. The set $M := \{x : \mathcal{D}_\omega^\pm f(x) = 1\}$ is totally disconnected. This can be proven by contradiction. Suppose that the set M is connected hence, $\mathcal{D}_\omega f(y)$ is continuous; and independently f is increasing in $I_y = [y - \epsilon, y + \epsilon] \subset M$. However, $(f(y) - f(y - \epsilon))(f(y + \epsilon) - f(y)) \leq 0$. Therefore, in the limit $\epsilon \to 0$

$$\mathcal{D}_\omega^- f(y) \mathcal{D}_\omega^+ f(y) \leq 0 \implies \mathcal{D}_\omega^- f(y) = -1.$$

However, this a contradiction since f was supposed to be increasing in I_y, hence M can not be connected.

By duality, the set $\{x : \mathcal{D}_\omega^\pm f(x) = -1\}$ is also totally disconnected. Hence, only the set $\{x : \mathcal{D}_\omega^\pm f(x) = 0\}$ must have connected components. $\square$

The same conclusion is true if we replace the indicial derivative with the supremum Dini-type derivative

Corollary 4. *Suppose that $f \cong \mathcal{C}[I]$ and $f \cong NM[I]$. Then*

$$\bar{\mathcal{D}}_\omega f(x) \cong \mathcal{C}[I] \implies \bar{\mathcal{D}}_\omega f(x) = 0$$

Theorem 10 (Continuity sets of derivatives). *Consider a continuous function f on a compact interval I. Suppose that $f'_+(x)$ and $f'_-(x)$ are separately continuous wherever they exist then the following holds:*

(1)　$f'_+(x) = f'_-(x) = f'(x)$
(2)　$\Delta_{f,I} := \{x : f' \notin \mathcal{C}, x \in I\}$ *is totally disconnected with empty interior.*
(3)　*The total discontinuity set can be written as $\Delta_{f,I} = \Delta_{1,f} \cup \Delta_{2,f}$, where $\Delta_{1,f}$ is F_σ and $\Delta_{2,f}$ is a null set.*
(4)　*The continuity set $\mathcal{C}_f$ is G_δ. F_σ and G_δ are given by Definition A4.*

Proof. Consider the interval $I = [u,v]$. Then there is rational $r \in \mathbb{Q} \cap I$.

Associate $(r, I) \equiv I_r$ so that I_r can be counted by an enumeration of the rationals.

Assume that $f'_+(x)$ and $f'_-(x)$ are separately continuous on the opening of $I_r^\circ = I_r \setminus \{u, v\}$. Fix x, such that $u \geq x > v$.

$$
\begin{aligned}
&\overset{u > v \;\; u \geq x > v}{\frac{f(u) - f(v)}{u - v} = \frac{f(u) - f(x) + f(x) - f(v)}{u - v}} = \\
&\frac{f(u) - f(x)}{u - x} \underbrace{\frac{u - x}{u - v}}_{1 - \lambda} + \frac{f(x) - f(v)}{x - v} \underbrace{\frac{x - v}{u - v}}_{\lambda} = \\
&\frac{f(u) - f(x)}{u - x}(1 - \lambda) + \frac{f(x) - f(v)}{x - v}\lambda \\
&\quad \downarrow \lim_{u \to x} \qquad\qquad \downarrow \lim_{v \to x} \\
&(1 - \lambda)f'_+(x) + \lambda f'_-(x) = f'_+(x) - \lambda\left(f'_+(x) - f'_-(x)\right)
\end{aligned}
$$

By continuity
$$\lim_{v \to x} f'_+(v) = f'_+(x) = f'_+(x) - \lambda\left(f'_+(x) - f'_-(x)\right)$$

However, since x and hence $\lambda \neq 0$ is arbitrary $f'_+(x) = f'_-(x)$ must hold $\forall x \in I_r^\circ$. Hence, f' is continuous on I_r°.

By this argument we establish that the set $\Delta_{1,f} := \{x : f' \ncong \mathcal{C}\} \cap I$ is F_σ, where we also assume that whenever $f'(x)$ does not exist it is replaced by a value that makes f' discontinuous. By Theorem 1 the discontinuity set is totally disconnected and with empty interior.

Let us further consider the case wherever left and right derivatives do not exist (either diverge or oscillate without a limit). It is enough to consider the right derivative. Then we have that for $0 < u, v \leq \delta$

$$\left| \frac{\Delta_u^+ [f](x)}{u} - \frac{\Delta_v^+ [f](x)}{v} \right| > \epsilon > 0 \tag{6}$$

for some ϵ. We can consider a variable $\xi \in [x, x + u] \cap [x, x + v] = [x, x + \min(u, v)] = J$. There is a rational $r \in \mathbb{Q} \cap J$. Associate $(r, J) \equiv J_r$ so that J_r can be counted by an enumeration of the rationals and index $\delta :: r$. Therefore, the set

$$\Delta_{2,f} := \bigcup_{k=1}^{\infty} \{z : \text{ineq (6) is true}, z \in J_k\}$$

is countable $\forall \delta > 0$. Since it is totally disconnected by Theorem 1 we can select $\delta_k = \delta / 2^k$. Therefore,

$$|\Delta_{2,f}| = \sum_{k=1}^{\infty} |J_k| \leq \sum_{k=1}^{\infty} \frac{\delta}{2^k} = \delta$$

Therefore, $\Delta_{2,f}$ is a null set.

The same argument can be applied to the left derivative considering $f(-x)$.

The total discontinuity set can be written as

$$\Delta_{f,I} = \Delta_{1,f} \cup \Delta_{2,f}$$

Therefore, the continuity set can be written as $\mathcal{C}_f = \left(\Delta_{1,f} \cup \Delta_{2,f} \right)^c$, hence it is a G_δ set. □

7. Characterization of Fractional Velocities

The notion of fractional velocity was introduced by Cherbit [10] independently from the earlier authors. His initial idea was to establish a link between the fractional dimension of a curve and its Hölder exponent in a local manner. This amounted to replacing the linear increment in the denominator of the difference quotient with an increment raised to a fractional power. What was not recognized at that time was the inherent asymmetry of such definition, which lead to the misplaced expectation that the left and right velocities should be equal. Fractional velocities were used to characterize the growth of some singular functions [5]. Therefore, it is of interests to establish its relationship with the indicial derivative.

Definition 14. *Define Fractional Variation operators of order $0 \leq \beta \leq 1$ as*

$$v_\beta^{\epsilon \pm} [f] (x) := \frac{\Delta_\epsilon^{\pm} [f] (x)}{\epsilon^\beta} \tag{7}$$

for a positive ϵ [14].

The notation for fractional variation is similar to the one used for fractional velocity in order to emphasize the fact that the latter concept is derived from the former.

Definition 15 (Fractional order velocity)**.** *Define the fractional velocity [11] of fractional order β as the limit*

$$v_\pm^\beta f (x) := \lim_{\epsilon \to 0} v_\beta^{\epsilon \pm} [f] (x) \tag{8}$$

A function for which at least one of $v_\pm^\beta f (x)$ exists finitely will be called β-differentiable at the point x.

Remark 2. *In the above definition, we do not require upfront equality of left and right β-velocities. The fractional velocities are related to the ordinary (i.e., Cauchy) derivatives of locally continuous functions as*

$$v_\pm^\beta f (x) = \pm \frac{1}{\beta} \lim_{\epsilon \to 0} \epsilon^{1-\beta} f(x \pm \epsilon)$$

recognized at first in [14]. Because of this relationship the term velocity is warranted for applications.

On the other hand, classical fractional derivatives (i.e., Riemann-Liouville) are non-local convolution operators. Therefore, physically such fractional derivative can not be interpreted as a rate of change, but can be related to the memory of the process about past changes.

Definition 16. *The set of points where the fractional velocity exists finitely and does not vanish will be denoted as the "set of change":*

$$\chi_{\pm}^{\beta}(f) := \left\{ x : v_{\pm}^{\beta} f(x) \neq 0 \right\}$$

Proposition 1. *Suppose that $x \in \chi_{\pm}^{\beta}(f)$ Then*

$$\mathcal{D}_{\omega}^{\pm} f(x) = K v_{\pm}^{\beta} f(x)$$

for some constant $K > 0$.

Proof. Consider the forward fractional velocity. Since $x \in \chi_{+}^{\beta}(f)$ then the limit

$$K = \lim_{\epsilon \to 0} \frac{\omega_x^+(\epsilon)}{\epsilon^{\beta}}$$

exists as a finite positive number. Then

$$\frac{\Delta_{\epsilon}^+ [f](x)}{\epsilon^{\beta}} = \frac{\Delta_{\epsilon}^+ [f](x)}{\omega_x^+(\epsilon)} \frac{\omega_x^+(\epsilon)}{\epsilon^{\beta}}$$

Taking the limit both sides results in

$$v_{+}^{\beta} f(x) = \lim_{\epsilon \to 0} \frac{\Delta_{\epsilon}^+ [f](x)}{\omega_x^+(\epsilon)} K$$

Therefore, $\mathcal{D}_{\omega}^+ f(x)$ exists. The same argument applies to the backward fractional velocity. □

8. Discussion

The presented results can be discussed in several directions. Theorem 1 has merits of its own and can point towards an interesting avenue for future research. In particular, it may be instructive to pursue its generalizations in $\mathbb{R}^n$ or more abstract topological spaces.

The Lebesgue differentiation theorem is an important mathematical tool used for proving results also for functions of bounded variation. The results of the present paper demonstrate that caution is needed when applying the theorem to quasi-singular functions. The theorem can still be applied if the derivative is interpreted as the one-sided derivative in view of Theorem 10.

The fractional velocity can be interpreted as the rate of change with regard a power law, while an indicial derivative can be interpreted as the rate of change with regard to the magnitude of oscillation. This is more abstract than any given power law and focuses the application to more topological questions as demonstrated in the present work.

In contrast to fractional velocities [11], the indicial derivatives have connected sets of change. This puts them in a category, which is akin to the usual, Cauchy derivatives. Therefore, it can be instructive to demonstrate under what conditions the set of change is totally disconnected and if this happens only if the point-wise Hölder exponent is fractional.

Another line of research could be the application of the indicial derivatives to random objects, such as the paths of the Wiener process. This could be a model of constrained random walk with reflection.

Theorem 6 represents the best possible result for the non-differentiability set of local-type of derivatives and partially corresponds to the expectation of Ampere.

From a physical perspective, the quasi-singular functions, such as the one demonstrated here, could be also interesting objects of study. For example, these can represent the distribution of defects in a material.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. Additional Notations

The next definition is given by Bartle [15] [Part 1, Chape 7, p. 103].

Definition A1 (Function of Bounded Variation). *Consider the interval $I = [a, b]$. A partition of I is a set of $n + 1$ numbers $\mathcal{P}[I] := (a < x_1 \ldots x_{n-1} < b)$. The function $f : \mathbb{R} \mapsto \mathbb{R}$ is said to be of bounded variation on I if and only if there is a constant $M > 0$ such that*

$$V_{\mathcal{P}}[I] := (\mathcal{P}) \sum_{i=1}^{n+1} |f(x_i - x_{i-1})| \leq M$$

The total variation of the function is defined as

$$Var(f, I) := \sup_{\mathcal{P}} V_{\mathcal{P}}[I]$$

where the supremum is taken in all partitions $\mathcal{P}$. The class of function of bonded variation in a compact interval I will be denoted as $BV[I]$.

Definition A2 (Null sets). *A **null set** $Z \subset \mathbb{R}$ (or a set of measure 0) is called a set, such that for every $0 < \epsilon < 1$ there is a countable collection of sub-intervals $\{I_k\}_{k=1}^{\infty}$, such that*

$$Z \subseteq \bigcup_{k=1}^{\infty} I_k, \quad \sum_{k=1}^{\infty} |I_k| \leq \epsilon$$

where $|.|$ is the interval length. Then we write $|Z| = 0$.

Definition A3 (Baire function classes). *The function $f : \mathbb{R} \mapsto \mathbb{R}$ is called Baire-class I if there is a sequence of continuous functions converging to f point-wise.*

Definition A4 (G_δ and F_σ sets). *Let X be a metric space.*

- *The set $E \subseteq X$ is G_δ if it is countable intersection of open sets, and it is F_σ if it is countable union of closed sets.*
- *The set $E \subseteq X$ is meagre if it can be expressed as the union of countably many nowhere dense subsets of X.*
- *Dually, a co-meagre set is one whose complement is meagre, or equivalently, the intersection of countably many sets with dense interiors.*

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