

SCROLLAR INVARIANTS, SYZYGIES AND REPRESENTATIONS OF THE SYMMETRIC GROUP

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ABSTRACT. We give an explicit minimal graded free resolution, in terms of representations of the symmetric group S_d , of a Galois-theoretic configuration of d points in \mathbf{P}^{d-2} that was studied by Bhargava in the context of ring parametrizations. When applied to the geometric generic fiber of a simply branched degree d cover of \mathbf{P}^1 by a relatively canonically embedded curve C , our construction gives a new interpretation for the splitting types of the syzygy bundles appearing in its relative minimal resolution. Concretely, our work implies that all these splitting types consist of scollar invariants of resolvent covers. This vastly generalizes a prior observation due to Casnati, namely that the first syzygy bundle of a degree 4 cover splits according to the scollar invariants of its cubic resolvent. Our work also shows that the splitting types of the syzygy bundles, together with the multi-set of scollar invariants, belong to a much larger class of multi-sets of invariants that can be attached to $C \rightarrow \mathbf{P}^1$: one for each irreducible representation of S_d , i.e., one for each partition of d .

1. INTRODUCTION

(1.1). This article, which is an extended version of [16], is concerned with the “scollar invariants” of a curve C with respect to a separable morphism $\varphi : C \rightarrow \mathbf{P}^1$. Throughout, all curves are assumed to be smooth, projective and geometrically integral, unless otherwise stated. We recall, e.g. from [18, §1.2], that the scollar invariants of C with respect to φ are the positive integers $e_1 \leq e_2 \leq \dots \leq e_{d-1}$ for which

$$(1) \quad \varphi_* \mathcal{O}_C \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-e_1) \oplus \dots \oplus \mathcal{O}_{\mathbf{P}^1}(-e_{d-1}),$$

where d denotes the degree of φ . Some prefer the equivalent characterization of e_i as the minimal n for which $h^0(C, nD) - h^0(C, (n-1)D) > i$, with D any geometric fiber of φ . The scollar invariants sum up to $g + d - 1$, with g the genus of C , and they satisfy $e_{d-1} \leq (2g + 2d - 2)/d$; this upper bound will be referred to as the “Maroni bound”. As a side remark, let us point out that the scollar invariants e_i are the function-field analogues of $\log \lambda_i$, with $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{d-1}$ the non-trivial successive minima of the Minkowski lattice attached to a degree d number field [30, §7]; thus, studying the scollar invariants of C with respect to φ is closely related to

studying the “geometry” of the corresponding function field extension, in Minkowski’s sense.

(1.2). We caution the reader for an ambiguity in the existing literature: several references, in fact including [18, §1.2], define the scrollar invariants of C with respect to φ as the integers $e'_1 \leq e'_2 \leq \dots \leq e'_{d-1}$ for which $\varphi_* \mathcal{O}_C(K_C) \cong \mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1}(e'_1) \oplus \dots \oplus \mathcal{O}_{\mathbf{P}^1}(e'_{d-1})$, with K_C some canonical divisor on C . The Riemann–Roch theorem implies that $e'_i = e_i - 2$ for all i .¹ Consequently, when interpreting our results for this alternative definition, the shift by -2 must be taken into account.

(1.3). **Contributions.** Consider a degree d cover $\varphi : C \rightarrow \mathbf{P}^1$ over a field k with $\text{char } k = 0$ or $\text{char } k > d$, and assume for technical convenience that φ is simply branched, i.e., geometrically, all non-trivial ramification is of type $(2, 1^{d-2})$. This ensures that the Galois closure

$$\bar{\varphi} : \bar{C} \rightarrow C \xrightarrow{\varphi} \mathbf{P}^1$$

has the full symmetric group S_d as its Galois group over \mathbf{P}^1 [26, Lem. 6.10]. By the normal basis theorem we can view $L = k(\bar{C})$ as the regular representation of S_d over $k(t) = k(\mathbf{P}^1)$, and its decomposition $L = \bigoplus_{\lambda \vdash d} W_\lambda$ into isotypic components induces a decomposition

$$(2) \quad \bar{\varphi}_* \mathcal{O}_{\bar{C}} \cong \bigoplus_{\lambda \vdash d} \mathcal{W}_\lambda$$

into vector bundles \mathcal{W}_λ of rank $(\dim V_\lambda)^2$, where V_λ denotes the irreducible representation (i.e., the Specht module) corresponding to the partition λ . As will be explained in Section 4, we can further decompose \mathcal{W}_λ as

$$(3) \quad \begin{array}{ccccccc} \mathcal{O}_{\mathbf{P}^1}(-e_{\lambda,1}) & \oplus & \mathcal{O}_{\mathbf{P}^1}(-e_{\lambda,2}) & \oplus & \dots & \oplus & \mathcal{O}_{\mathbf{P}^1}(-e_{\lambda, \dim V_\lambda}) \\ \mathcal{O}_{\mathbf{P}^1}(-e_{\lambda,1}) & \oplus & \mathcal{O}_{\mathbf{P}^1}(-e_{\lambda,2}) & \oplus & \dots & \oplus & \mathcal{O}_{\mathbf{P}^1}(-e_{\lambda, \dim V_\lambda}) \\ \vdots & & \vdots & & \ddots & & \vdots \\ \mathcal{O}_{\mathbf{P}^1}(-e_{\lambda,1}) & \oplus & \mathcal{O}_{\mathbf{P}^1}(-e_{\lambda,2}) & \oplus & \dots & \oplus & \mathcal{O}_{\mathbf{P}^1}(-e_{\lambda, \dim V_\lambda}), \end{array}$$

where every column (i.e., every “vertical slice”) contains $\dim V_\lambda$ copies of the same entry. Our main objects of study are the integers obtained by selecting a “horizontal slice”:

Definition 1. Under the above notation and assumptions, we call $\{e_{\lambda,1}, \dots, e_{\lambda, \dim V_\lambda}\}$ the multi-set of “scrollar invariants of λ with respect to φ ”.

One sees that the $d! - 1$ scrollar invariants of \bar{C} with respect to $\bar{\varphi}$ are obtained by taking the union, over all non-trivial partitions $\lambda \vdash d$, of the multi-sets of scrollar invariants of λ with respect to φ , where each multi-set is to be considered with multiplicity $\dim V_\lambda$.

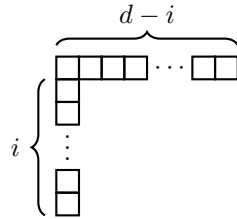
Definition 1 generalizes the notion of scrollar invariants of C with respect to φ . Indeed, as a consequence to Proposition 2 below, we recover

¹See also Footnote 10.

$\{e_1, e_2, \dots, e_{d-1}\}$ as the multi-set of scollar invariants of the partition $(d-1, 1)$ with respect to φ . Some basic properties generalize as well: e.g., in (4.9) we will prove a “volume formula” for the sum of the scollar invariants with respect to any partition $\lambda \vdash d$, thereby generalizing the identity $e_1 + e_2 + \dots + e_{d-1} = g + d - 1$. We will also prove a duality statement relating the scollar invariants with respect to λ to those with respect to the dual partition λ^* (i.e., the partition obtained by transposing its Young diagram).

(1.4). We remark that these generalized scollar invariants have appeared before, at least implicitly. Indeed, they describe the splitting types of the underlying vector bundle \mathcal{E}_λ of the parabolic bundle attached to V_λ under the Mehta–Seshadri correspondence [38], where V_λ is viewed as a representation of $\pi_1^{\text{geom}}(\mathbf{P}^1 \setminus \text{branch locus of } \varphi)$ via its natural map to $\text{Gal}(C/\mathbf{P}^1) \cong S_d$. Equivalently, one finds \mathcal{E}_λ as the Deligne canonical extension to \mathbf{P}^1 of the local system attached to this representation. We have $\mathcal{W}_\lambda \cong \mathcal{E}_\lambda^{\dim V_\lambda}$. The reader is forwarded to the recent works by Landesman–Litt [34, §2], [35, §3] and the references therein for further details.

(1.5). For certain partitions $\lambda \vdash d$, we managed to relate the corresponding multi-sets of scollar invariants to known data. The easiest cases are the hooks, with Young diagrams



for $i = 0, 1, \dots, d-1$. Concretely, in Section 4 we will prove:

Proposition 2. *Consider a simply branched degree $d \geq 2$ cover $\varphi : C \rightarrow \mathbf{P}^1$ over a field k with $\text{char } k = 0$ or $\text{char } k > d$ with scollar invariants e_1, e_2, \dots, e_{d-1} , and let $i \in \{0, 1, \dots, d-1\}$. The multi-set of scollar invariants of the partition $(d-i, 1^i)$ with respect to φ is*

$$\left\{ \sum_{\ell \in S} e_\ell \mid S \text{ is an } i\text{-element subset of } \{1, 2, \dots, d-1\} \right\}.$$

For $i = 1$, corresponding to the standard representation $V_{(d-1,1)}$, we indeed recover the scollar invariants of C with respect to φ . For $i = 0$, corresponding to the trivial representation $V_{(d)}$, we find the unique scollar invariant 0. For $i = d-1$, corresponding to the sign representation $V_{(1^d)}$, we find the unique scollar invariant $e_1 + e_2 + \dots + e_{d-1} = g + d - 1$.

(1.6). The main result of this article is a concrete and surprising interpretation for the multi-set of scollar invariants of the partition $\lambda_{i+1} = (d - i - 1, 2, 1^{i-1})$

$$(4) \quad \begin{array}{c} \overbrace{\begin{array}{cccc} \square & \square & \square & \square \end{array}}^{d-i-1} \\ \begin{array}{c} \square \\ \square \\ \vdots \\ \square \end{array} \end{array}$$

for any $i = 1, 2, \dots, d - 3$: in Section 5 we show that this multi-set equals the splitting type

$$(5) \quad \left\{ b_j^{(i)} \mid j = 1, \dots, \beta_i \right\} \quad \text{with } \beta_i = \frac{d}{i+1}(d-2-i) \binom{d-2}{i-1}$$

of the i th syzygy bundle in the relative canonical resolution of C with respect to φ , as introduced by Casnati–Ekedahl [12], who built on work of Schreyer [45]; see (1.10) below for more details. We will occasionally refer to the elements of this splitting type as “Schreyer invariants of C with respect to φ ”.

Theorem 3. *Consider a simply branched degree $d \geq 4$ cover $\varphi : C \rightarrow \mathbf{P}^1$ over a field k with $\text{char } k = 0$ or $\text{char } k > d$, and let $i \in \{1, \dots, d - 3\}$. The multi-set of scollar invariants of the partition $(d - i - 1, 2, 1^{i-1})$ with respect to φ is equal to the splitting type of the i th syzygy bundle of the relative canonical resolution of C with respect to φ .*

Symbolically: for $\lambda = \lambda_{i+1}$ the partition from (4) we have

$$e_{\lambda,j} = b_j^{(i)}, \quad j = 1, 2, \dots, \dim V_{(d-i-1, 2, 1^{i-1})}$$

for a suitable ordering of the $e_{\lambda,j}$ ’s. As a sanity check, the reader is invited to verify that $\dim V_{(d-i-1, 2, 1^{i-1})}$ indeed equals β_i , using the hook length formula. In (4.13) we will slightly relax the simple branching assumption to the condition that all non-trivial ramification is of the form $(2, 1^{d-2})$ or $(3, 1^{d-3})$.

As a consequence to Theorem 3 we find that the splitting types of the syzygy bundles turn out to consist of scollar invariants, namely of the Galois closure $\overline{\varphi} : \overline{C} \rightarrow \mathbf{P}^1$. Thus, they too are “geometric” in Minkowski’s sense. We know of one prior observation of this kind: for $d = 4$ it was noted that the scollar invariants of the degree 3 cover obtained from φ through Recillas’ trigonal construction are given by b_1 and b_2 , with $\{b_1, b_2\}$ the splitting type of the first syzygy bundle of C with respect to φ . This observation seems due to Casnati [11, Def. 6.3-6.4], although we refer to Deopurkar–Patel [21, Prop. 4.6] for a more explicit mention. As will become clear, our main result can be viewed as a vast generalization of this.

Beyond the families $(d-i, 1^i)$ and $(d-i-1, 2, 1^{i-1})$, we did not succeed in finding partitions whose corresponding scrollar invariants can be related to known data and we in fact believe that they are all genuinely new. The first such partitions appear in degree $d = 6$, namely (2^3) and (3^2) , corresponding to invariants a_1, a_2, \dots, a_5 and their duals $g + 5 - a_1, g + 5 - a_2, \dots, g + 5 - a_5$, which seem unrelated to both the scrollar invariants and the Schreyer invariants.

(1.7). For any given subgroup $H \subseteq S_d$, we can look at the subfield $L^H \subseteq L$ that is fixed by H . The corresponding degree $[S_d : H] = d!/|H|$ covering

$$R_H \varphi : R_H C \rightarrow \mathbf{P}^1$$

is called the “resolvent of φ with respect to H ”. For $H = \{\text{id}\}$ we recover the Galois closure $\overline{\varphi} : \overline{C} \rightarrow \mathbf{P}^1$: recall that its scrollar invariants are obtained by taking the union of the multi-sets of scrollar invariants with respect to λ , over all non-trivial partitions λ , where each multi-set is to be considered with multiplicity $\dim V_\lambda$. For general H , this remains true but the multiplicities change:

Theorem 4. *Consider a simply branched degree $d \geq 2$ cover $\varphi : C \rightarrow \mathbf{P}^1$ over a field k with $\text{char } k = 0$ or $\text{char } k > d$, along with a subgroup $H \subseteq S_d$. The scrollar invariants of $R_H C$ with respect to $R_H \varphi$ are found by taking the union, over all non-trivial partitions $\lambda \vdash d$, of the multi-sets of scrollar invariants of λ with respect to φ , where each multi-set is to be considered with multiplicity*

$$\text{mult}(V_\lambda, \text{Ind}_H^{S_d} \mathbf{1}).$$

Here $\text{Ind}_H^{S_d} \mathbf{1}$ denotes the representation of S_d that is induced by the trivial representation of H (i.e., it is the permutation representation of S_d/H).

In other words, the decomposition of $(R_H \varphi)_* \mathcal{O}_{R_H C}$ is obtained by taking $\text{mult}(V_\lambda, \text{Ind}_H^{S_d} \mathbf{1})$ horizontal slices of the block (3) corresponding to λ , for each partition $\lambda \vdash d$. A proof can be found in Section 4. Note that some multiplicities may be zero, in which case the corresponding scrollar invariants do not appear. E.g., for $d = 4$ and $D_4 = \langle (12), (1324) \rangle$ the dihedral group of order 8 one has $\text{Ind}_{D_4}^{S_4} \mathbf{1} \cong V_{(4)} \oplus V_{(2^2)}$, which in combination with Theorem 3 shows that the scrollar invariants of $R_{D_4} C$ with respect to $R_{D_4} \varphi$ are given by b_1 and b_2 . This is not a coincidence, as the resolvent with respect to D_4 (also known as “Lagrange’s cubic resolvent”) is nothing but the degree 3 covering found through Recillas’ trigonal construction [52, §8.6].

As another exemplary corollary to Theorems 3 and 4 we state:

Theorem 5. *Consider a simply branched degree $d \geq 4$ cover $\varphi : C \rightarrow \mathbf{P}^1$ over a field k with $\text{char } k = 0$ or $\text{char } k > d$. Let H be the Young subgroup $S_2 \times S_{d-2}$ of S_d . Then the multi-set of scrollar invariants of $R_H C$ with respect to $R_H \varphi$ is obtained by taking the union of the multi-sets*

- $\{e_1, e_2, \dots, e_{d-1}\}$, the scrollar invariants of C with respect to φ , and

- $\{b_1, b_2, \dots, b_{d(d-3)/2}\}$, the splitting type of the first syzygy bundle of C with respect to φ .

The (very short) proof can be read in (5.7).

(1.8). **Syzygies from Galois representations.** The main auxiliary tool behind Theorem 3 is an explicit, purely Galois-theoretic way of constructing minimal graded free resolutions of certain configurations of $d \geq 4$ points in \mathbf{P}^{d-2} , that were introduced by Bhargava in the context of ring parametrizations [4, §2]. These configurations arise by considering a degree d extension $F \subseteq K$ of fields with $\text{char } F = 0$ or $\text{char } F > d$, along with a basis $\alpha_0 = 1, \alpha_1, \dots, \alpha_{d-1}$ of K over F . Denote by L the Galois closure; we assume for convenience that $\text{Gal}(L/F)$ is the full symmetric group S_d . Write $\sigma_1 = \text{id}, \sigma_2, \dots, \sigma_d$ for the embeddings $K \hookrightarrow L$ that fix F element-wise, take the dual basis $\alpha_0^*, \alpha_1^*, \dots, \alpha_{d-1}^*$ with respect to $\text{Tr}_{L/F}$, and define

$$\alpha_i^{*(j)} = \sigma_j(\alpha_i^*)$$

for each $i = 0, 1, \dots, d-1$ and $j = 1, \dots, d$. Then the requested points in \mathbf{P}^{d-2} are

$$(6) \quad [\alpha_1^{*(1)} : \dots : \alpha_{d-1}^{*(1)}], [\alpha_1^{*(2)} : \dots : \alpha_{d-1}^{*(2)}], \dots, [\alpha_1^{*(d)} : \dots : \alpha_{d-1}^{*(d)}].$$

No $d-1$ of these points lie on a hyperplane, so they are “in general position”.

Thus, from [45, (4.2)] we know that any minimal graded free resolution of their joint coordinate ring must have

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & \dots & d-3 & d-4 \\ \hline 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \beta_1 & \beta_2 & \dots & \beta_{d-3} & 0 \\ 2 & 0 & 0 & 0 & \dots & 0 & 1 \end{array}$$

as its Betti table. In our minimal graded free resolution, the details of which can be found in Section 3, the i th syzygy module arises from the isotypic subrepresentation $W_{\lambda_{i+1}} \subseteq L$ corresponding to the partition $\lambda_{i+1} = (d-i-1, 2, 1^{i-1})$, where as before we view L as the regular representation of S_d through the normal basis theorem.

The connection between the partitions λ_{i+1} and syzygies of d general points in \mathbf{P}^{d-2} is not a new observation: this was studied by Wilson [56, §5]. In his discussion, the syzygy modules are constructed from the Specht modules $V_{\lambda_{i+1}}$; we tend to think of these as “vertical slices” of the corresponding isotypic components $W_{\lambda_{i+1}}$. Our new Galois-theoretic construction is somehow orthogonal to this and uses “horizontal slices”, which are not representations. Nonetheless, as we will see, they better suit our needs.

(1.9). In Section 5 we will explain how Bhargava’s point configuration shows up very naturally when studying the geometric generic fiber of our

cover $\varphi : C \rightarrow \mathbf{P}^1$, henceforth assumed to be of degree $d \geq 4$. In more detail, from Casnati–Ekedahl [12, Thm. 2.1] we know that φ decomposes as

$$C \xrightarrow{\iota} \mathbf{P}(\mathcal{E}) \xrightarrow{\pi} \mathbf{P}^1, \quad \mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \dots \oplus \mathcal{O}_{\mathbf{P}^1}(e_{d-1}),$$

with π the natural \mathbf{P}^{d-2} -bundle map and ι the “relative canonical embedding”.² By identifying C with its relative canonical image, we can view each geometric fiber of φ , including the geometric generic fiber, as a configuration of d points in \mathbf{P}^{d-2} ; an illustration of the case $d = 4$ can be found in Figure 1. The field of definition of any point in the support of the geomet-

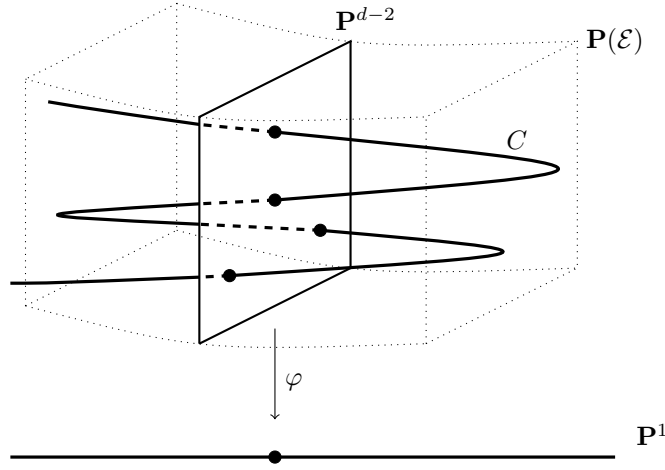


FIGURE 1. Configuration of d points in \mathbf{P}^{d-2} attached to φ .

ric generic fiber can be identified with $k(C)$. It is then always possible to equip that point with projective coordinates $[\alpha_1^* : \alpha_2^* : \dots : \alpha_{d-1}^*]$ satisfying $\text{Tr}_{k(C)/k(t)}(\alpha_i^*) = 0$ for all $i = 1, \dots, d-1$. By dualizing, we obtain a basis $1, \alpha_1, \dots, \alpha_{d-1}$ of $k(C)$ over $k(t)$ from which the geometric generic fiber of φ is recovered through Bhargava’s construction. (We will actually need a slightly stronger fact, namely that $1, \alpha_1, \dots, \alpha_{d-1}$ can be arranged to be a so-called “reduced basis” of $k(C)$ over $k(t)$; see (5.3) for further details.)

²The standard (i.e., absolute) canonical map $C \rightarrow \mathbf{P}^{g-1}$ is obtained from ι by composing it with the “tautological map”

$$\kappa : \mathbf{P}(\mathcal{E}) \xrightarrow{\cong} \mathbf{P}(\mathcal{E}(-2)) \rightarrow \mathbf{P}^{e_1 + \dots + e_{d-1} - d} = \mathbf{P}^{g-1},$$

the image of which is a rational normal scroll; see [25, §1]. If $e_1 > 2$ then κ is an embedding, in which case one can reverse the construction and recover $\mathbf{P}(\mathcal{E})$ from the canonical model of C as the union of the linear spans inside \mathbf{P}^{g-1} of the fibers of φ . Each such linear span is indeed a \mathbf{P}^{d-2} , by the geometric Riemann–Roch theorem. This is Schreyer’s original approach from [45].

(1.10). Casnati–Ekedahl, building on Schreyer, further showed that a minimal graded free resolution of the geometric generic fiber can be completed to a minimal resolution of C relative to $\mathbf{P}(\mathcal{E})$:

$$(7) \quad 0 \rightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-dH + (g - d - 1)R) \rightarrow \\ \bigoplus_{j=1}^{\beta_{d-3}} \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-(d-2)H + b_j^{(d-3)}R) \rightarrow \bigoplus_{j=1}^{\beta_{d-4}} \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-(d-3)H + b_j^{(d-4)}R) \rightarrow \\ \cdots \rightarrow \bigoplus_{j=1}^{\beta_1} \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-2H + b_j^{(1)}R) \rightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})} \rightarrow \mathcal{O}_C \rightarrow 0.$$

Here $R = [\pi^* \mathcal{O}_{\mathbf{P}^1}(1)]$ and $H = [\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)]$ denote the \mathbf{P}^{d-2} -ruling and the class of “hyperplane sections”,³ respectively, known to form a \mathbf{Z} -basis of $\text{Pic}(\mathbf{P}(\mathcal{E}))$; see [25, 45]. This introduces the Schreyer invariants from (5). At this point, it should come as no surprise to the reader that our strategy to prove Theorem 3 will be to apply this construction to the minimal graded free resolution discussed in (1.8).

(1.11). Recall from (1.2) that the scrollar invariants are being defined inconsistently in the existing literature. There is a corresponding ambiguity for the Schreyer invariants: several references, including Schreyer’s original treatment [45], instead define them as

$$b_i^{(j)} - 2i - 2.$$

The reason for the shifts is that working with $\mathcal{E}(-2)$ makes it more natural to use the generator $H - 2R \in \text{Pic}(\mathbf{P}(\mathcal{E}))$ rather than H . Thus, when interpreting our results for the invariants as they were initially introduced by Schreyer, the shifts must be taken into account.

(1.12). **Further applications and remarks.** Our work has three immediate further applications, which are discussed in Section 7. Firstly, Theorem 4 in combination with Proposition 2 and Theorem 3 gives a way of constructing many new examples of multi-sets of integers that are realizable as the multi-set of scrollar invariants of some \mathbf{P}^1 -cover. Most of these examples are highly non-balanced, i.e., there are large gaps between the scrollar invariants, so they live in a different regime from the ones provided by the existing literature, see e.g. [1, 14, 17, 19]. Secondly, since the Schreyer invariants are scrollar, they are subject to the “Maroni bound” coming from the Riemann–Roch theorem, which leads to non-trivial upper and lower bounds that seem unreported. Thirdly, from Theorem 4 we see that Gassmann equivalent subgroups of S_d (see (7.6)) give rise to resolvent covers having

³More precisely: $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) = j^* \mathcal{O}_{\mathbf{P}^{g+2d-3}}(1)$ with $j : \mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}^{e_1+\dots+e_{d-1}+d-2}$ the “tautological map” from [25, §1], but now associated with \mathcal{E} rather than with $\mathcal{E}(-2)$ as was the case in Footnote 2 (this difference is the source of the ambiguity mentioned in (1.11)).

coinciding multi-sets of scrollar invariants. The number-theoretic counterpart of this statement reads that arithmetically equivalent number fields have Minkowski lattices with similar-sized successive minima: this fact was recently proved by the second-listed author [55], taking inspiration from the current work.

(1.13). A multi-set Σ of integers is “balanced” if $|\max \Sigma - \min \Sigma| \leq 1$. Consider the Hurwitz space $\mathcal{H}_{d,g}$ of simply branched degree d covers $\varphi : C \rightarrow \mathbf{P}^1$ by curves of genus g . Ballico [1] has proved that the multi-set of scrollar invariants of a sufficiently general element of $\mathcal{H}_{d,g}$ is balanced. Bujokas and Patel [10, Conj. A] have conjectured that the same is true for the splitting types of the syzygy bundles of a relative minimal resolution, provided that g is sufficiently large with respect to d .⁴ They prove this for the first syzygy bundle subject to the bound $g \geq (d-2)^2$, and for all higher syzygy bundles assuming $g \equiv 1 \pmod d$.

More generally, for any partition $\lambda \vdash d$ one can wonder about the generic behaviour of its scrollar invariants on $\mathcal{H}_{d,g}$. By Proposition 2 one cannot expect balancedness for all λ , even if g is large enough.⁵ We ask:

Problem 6. *Let $d \geq 2$ and $g \geq 0$ be integers and let $\lambda \vdash d$ be a non-trivial partition. Consider an algebraically closed field k with $\text{char } k = 0$ or $\text{char } k > d$. Under what conditions on d, g, λ can we conclude that the scrollar invariants of λ with respect to a general element $\varphi : C \rightarrow \mathbf{P}^1$ of $\mathcal{H}_{d,g}$ over k are balanced?*

A weaker question is answered in a recent preprint by Landesman and Litt [35, Ex. 1.3.7], who prove that the scrollar invariants of any $\lambda \vdash d$ with respect to a sufficiently general element $\varphi : C \rightarrow \mathbf{P}^1$ of $\mathcal{H}_{d,g}$ are always “consecutive”: if $e_{\lambda,1} \leq \dots \leq e_{\lambda, \dim V_\lambda}$ are the scrollar invariants of λ with respect to φ then $e_{\lambda,i+1} - e_{\lambda,i} \leq 1$ for $i = 1, \dots, \dim V_\lambda - 1$.

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⁴Work by Bopp and Hoff [8] shows the necessity of this assumption.

⁵We thank Aaron Landesman for pointing this out to us.

2. AUXILIARY FACTS FROM REPRESENTATION THEORY

(2.1). We begin with some notation. Fix an integer $d \geq 4$. For a partition $\lambda = (d_r, \dots, d_1)$ of d we interchangeably write

$$S_{d_1} \times S_{d_2} \times \cdots \times S_{d_r} \quad \text{and} \quad S_\lambda$$

for the Young subgroup $\text{Sym}\{1, \dots, d_1\} \times \text{Sym}\{d_1+1, \dots, d_2\} \times \cdots \times \text{Sym}\{d-d_r+1, \dots, d\}$ of $S_d = \text{Sym}\{1, \dots, d\}$, obtained by concatenating cycles. If no confusion is possible, then for $d' < d$ we will view $S_{d'}$ as a subgroup of S_d by identifying it with $S_1 \times S_1 \times \cdots \times S_1 \times S_{d'}$. For $i \in \{2, \dots, d-2\}$ we write λ_i to denote the partition $(d-i, 2, 1^{i-2})$. We extend this notation by letting $\lambda_0 = (d)$, $\lambda_1 = (d-1, 1)$ and $\lambda_d = (1^d)$. Note that we do not define λ_{d-1} . If R is a \mathbf{Z} -graded ring (e.g., a polynomial ring), then for any $i \in \mathbf{Z}$ we write R_i to denote its homogeneous degree i part.

(2.2). We state two basic facts on representations of finite groups, where we work over an arbitrary field F with $\text{char } F = 0$ or $\text{char } F > d$. We stick to S_d , but modulo further assumptions on F (being algebraically closed of characteristic 0 is always sufficient) the direct generalizations of these results hold for any finite group and are well-known to specialists.

Lemma 7. *Let V be an irreducible representation of S_d and let $H \subseteq S_d$ be a subgroup. Then $\dim V^H = \text{mult}(V, \text{Ind}_H^{S_d} \mathbf{1})$.*

Proof. We have

$$\begin{aligned} \dim V^H &= \text{mult}(\mathbf{1}, \text{Res}_H^{S_d} V) = \dim \text{Hom}_H(\mathbf{1}, \text{Res}_H^{S_d} V) \\ &= \dim \text{Hom}_{S_d}(\text{Ind}_H^{S_d} \mathbf{1}, V) = \text{mult}(V, \text{Ind}_H^{S_d} \mathbf{1}) \end{aligned}$$

where the third equality follows from Frobenius reciprocity. \square

Lemma 8. *Consider a partition $\lambda \vdash d$. Then there exists a unique $\rho \in Z(F[S_d])$ such that $\rho(V_\mu) = 0$ for every partition $\mu \neq \lambda$ and such that ρ induces the identity map on V_λ .*

(Here $Z(F[S_d])$ denotes the center of the group ring $F[S_d]$.)

Proof. This follows from [47, Thm. 8]. \square

(2.3). More specific to S_d , we have:

Lemma 9. *For any $2 \leq i \leq d-2$ we have $\dim(V_{\lambda_i}^{S_{\lambda_i}}) = 1$.*

Proof. We know that $\text{mult}(V_{\lambda_i}, \text{Ind}_{S_{\lambda_i}}^{S_d} \mathbf{1}) = 1$, see e.g. [44, Cor. 2.4.7], so this follows from Lemma 7. \square

We also need facts on how certain tensor products decompose into irreducibles:

Lemma 10. *Consider a partition $\lambda \vdash d$. Then*

$$V_\lambda \otimes V_{(d-1,1)} \cong \bigoplus_{\mu \vdash d} V_\mu^{c_{\mu\lambda} - \delta_{\mu\lambda}},$$

where $c_{\mu\lambda}$ equals the number of ways of transforming μ into λ by removing a box and adding a box; here $\delta_{\mu\lambda}$ denotes the Kronecker delta.

Proof. This can be found in [29, p. 257-258]. □

It is understood that the box removals are valid, in the sense that they result in the Young diagram of a partition of $d - 1$. Writing $\lambda = (d_r, \dots, d_1)$, we note that Lemma 10 admits the rephrasing

$$V_\lambda \otimes V_{(d-1,1)} \cong V_\lambda^{|\{d_1, \dots, d_r\}| - 1} \oplus \bigoplus_{\mu} V_\mu,$$

with μ ranging over all partitions whose Young diagram can be obtained from that of λ by removing one box and adding *another* box. Warning: here $|\{d_1, \dots, d_r\}|$ denotes the cardinality as a set, rather than as a multi-set.

Lemma 11. *Consider a partition $\lambda \vdash d$. Then*

$$V_\lambda \otimes V_{(d-2,2)} \cong \bigoplus_{\mu \vdash d} V_\mu^{-c_{\mu\lambda} + \frac{1}{2}(d_{\mu\lambda} + e_{\mu\lambda} - e'_{\mu\lambda})},$$

where $c_{\mu\lambda}$, $d_{\mu\lambda}$, $e_{\mu\lambda}$, $e'_{\mu\lambda}$ denote the number of ways of transforming μ into λ by

- removing a box and adding a box,
- consecutively removing two boxes and consecutively adding two boxes,
- removing two horizontally adjacent boxes and adding two horizontally adjacent boxes, or removing two vertically adjacent boxes and adding two vertically adjacent boxes,
- removing two horizontally adjacent boxes and adding two vertically adjacent boxes, or removing two vertically adjacent boxes and adding two horizontally adjacent boxes,

respectively.

Proof. This follows along [29, p. 258-259] or from a double application of Lemma 10. □

Using that

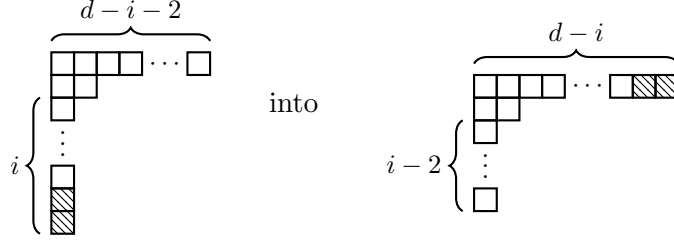
$$(8) \quad \text{Sym}^2 V_{(d-1,1)} \cong V_{(d)} \oplus V_{(d-1,1)} \oplus V_{(d-2,2)},$$

see [27, Prob. 4.19], we get the following corollaries:

Corollary 12. *We have*

$$\text{mult}(V_{\lambda_{i+2}}, V_{\lambda_i} \otimes \text{Sym}^2 V_{(d-1,1)}) = \begin{cases} 0 & \text{for all } 2 \leq i \leq d-4, \\ 1 & \text{if } i = d-2. \end{cases}$$

Proof. By Lemma 10 we only need to look at $V_{\lambda_{i+2}} \otimes V_{(d-2,2)}$, which we can handle with Lemma 11. Assume $2 \leq i \leq d-4$. Clearly $c_{\lambda_{i+2}\lambda_i} = 0$. There is only one way to transform



by consecutive removal of two boxes and consecutive addition of two boxes, so $d_{\lambda_{i+2}\lambda_i} = 1$. However, these boxes are vertically resp. horizontally adjacent, so $e_{\lambda_{i+2}\lambda_i} = 0$ and $e'_{\lambda_{i+2}\lambda_i} = 1$. The case $i = d-2$ follows similarly. \square

We also need particular statements about $\text{Sym}^3 V_{(d-1,1)}$ for which we give ad-hoc proofs:

Lemma 13. *Assume $d \geq 5$. Then*

$$\text{mult}(V_{\lambda_3}, \text{Sym}^3 V_{(d-1,1)}) = \text{mult}(V_{\lambda_d}, V_{\lambda_{d-3}} \otimes \text{Sym}^3 V_{(d-1,1)}) = 0.$$

Proof. As for the first vanishing, the Specht module V_{λ_3} can be naturally realized inside the polynomial ring $F[z_1, z_2, \dots, z_d]$ equipped with the natural S_d -action (where the z_i are variables), where it is generated by polynomials of degree $4 > 3$, see [27, Prob.4.47]. From [32, Prop.5] we see that V_{λ_3} cannot appear in the decomposition of $F[z_1, z_2, \dots, z_d]_3$, which contains $\text{Sym}^3 V_{(d-1,1)}$ as a subrepresentation, so the desired conclusion follows.

As for the second vanishing, tensoring $\text{Sym}^3 V_{(d-1,1)}$ with $V_{\lambda_{d-3}}$ produces the sign representation $V_{(1,\dots,1)}$ as a component if and only if tensoring with the dual $V_{(d-3,2,1)}$ of V_{λ_3} produces the trivial representation as a component. This can only happen if $V_{(d-3,2,1)}$ is a subrepresentation of $\text{Sym}^3 V_{(d-1,1)}$, which it is not, by the first part. \square

Finally, we will also make use of

$$(9) \quad \text{mult}(V_{(1^3)}, \text{Sym}^3 V_{(2,1)}) = 1 \quad \text{and} \quad \text{mult}(V_{(1^4)}, \text{Sym}^4 V_{(3,1)}) = 0,$$

which follow very easily along the previous lines of thought (or by explicit computation).

3. A MINIMAL FREE RESOLUTION FROM GALOIS THEORY

(3.1). We resume with the notation and assumptions from (1.8), where moreover we assume (without loss of generality) that $\text{Tr}_{L/F}(\alpha_i) = 0$ for $i = 1, \dots, d-1$. Note that Bhargava's point configuration (6) is closed under the action of $\text{Gal}(L/F)$. Therefore, as an algebraic set, it is defined over F . Write $R = F[x_1, \dots, x_{d-1}]$ and let $I \subseteq R$ be the ideal of this algebraic set. The main result of Section 3 is a new and explicit minimal free resolution of

the coordinate ring R/I as a graded R -module. As mentioned, our approach is Galois-theoretic and therefore quite different from that of Wilson [56, §5]. Instead, there is common ground with the approach of Behnke [2] and it may be possible to deduce several of the statements below from his work (we did not succeed in doing so). We note that this section is technical and the reader may want to skip it upon a first reading.

(3.2). Choosing α such that $K = F(\alpha)$, we can identify $\text{Gal}(L/F)$ with

$$\text{Sym} \left\{ \alpha^{(1)} = \sigma_1(\alpha) = \alpha, \alpha^{(2)} = \sigma_2(\alpha), \dots, \alpha^{(d)} = \sigma_d(\alpha) \right\},$$

which in turn is identified with $S_d = \text{Sym}\{1, 2, \dots, d\}$ by writing indices rather than field elements, e.g., $(1\ 2)$ refers to the field automorphism swapping $\alpha^{(1)}$ and $\alpha^{(2)}$ and fixing the other $\alpha^{(i)}$'s. As discussed in the introduction, we view L as the regular representation of S_d along with its isotypic components W_λ , with λ running over the partitions of d . For example, $W_{(d)}$ and $W_{(1^d)}$ are the one-dimensional subspaces generated by 1 and

$$\delta = \prod_{1 \leq i < j \leq d} (\alpha^{(i)} - \alpha^{(j)}),$$

respectively. A less degenerate example is $W_{(d-1,1)}$, which is the $(d-1)^2$ -dimensional subspace generated by the trace zero elements

$$\begin{array}{cccc} \alpha_1^{(1)}, & \alpha_2^{(1)}, & \cdots & \alpha_{d-1}^{(1)}, \\ \alpha_1^{(2)}, & \alpha_2^{(2)}, & \cdots & \alpha_{d-1}^{(2)}, \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{(d-1)}, & \alpha_2^{(d-1)}, & \cdots & \alpha_{d-1}^{(d-1)}, \end{array}$$

where $\alpha_i^{(j)} = \sigma_j(\alpha_i)$. Each of the vertical columns spans a $V_{(d-1,1)}$.

We fix a similar kind of basis for W_{λ_i} , for $i = 2, \dots, d-2$. Namely, from Lemma 9 we know that each irreducible subrepresentation of W_{λ_i} has a one-dimensional intersection with $L^{S_{\lambda_i}}$. Thus we can find linearly independent elements

$$\omega_1^i, \omega_2^i, \dots, \omega_{\beta_{i-1}}^i \in W_{\lambda_i}$$

that are fixed by S_{λ_i} . This leaves each ω_j^i with β_{i-1} linearly independent conjugates, which together span a V_{λ_i} . Taking the union over all j then produces our basis of W_{λ_i} .

(3.3). For $2 \leq i \leq d-2$ we consider the F -vector space V_i obtained from W_{λ_i} by intersecting it with $L^{S_{\lambda_i}}$, where λ_i is as defined in (2.1); in other words

$$V_i = \text{span}\{\omega_1^i, \omega_2^i, \dots, \omega_{\beta_{i-1}}^i\}.$$

We stress that these V_i 's are *not* subrepresentations of L . Instead, we think of V_i as a ‘‘horizontal slice’’ of our isotypic component W_{λ_i} , which can be recovered from V_i by taking its closure under the action of S_d (in other

words, W_{λ_i} is the smallest subrepresentation of L containing V_i). Likewise, we define $V_0 = F$ and $V_d = \text{span}\{\delta\} = F\delta$. Our resolution will take the form

$$\begin{aligned}
 (10) \quad & 0 \rightarrow V_d^* \otimes R(-d) \rightarrow V_{d-2}^* \otimes R(-d+2) \rightarrow V_{d-3}^* \otimes R(-d+3) \rightarrow \dots \\
 & \begin{array}{ccc}
 \begin{array}{c} \square \\ \square \\ \square \\ \vdots \\ \square \\ \square \\ \square \end{array} & \begin{array}{c} \square \square \\ \square \square \\ \square \\ \vdots \\ \square \\ \square \end{array} & \begin{array}{c} \square \square \square \\ \square \square \square \\ \square \square \\ \vdots \\ \square \end{array} \\
 & \rightarrow V_3^* \otimes R(-3) \rightarrow V_2^* \otimes R(-2) \rightarrow V_0^* \otimes R \rightarrow R/I \rightarrow 0, \\
 & \begin{array}{ccc}
 \begin{array}{c} \square \square \square \\ \square \square \end{array} \cdots \begin{array}{c} \square \square \end{array} & \begin{array}{c} \square \square \end{array} \cdots \begin{array}{c} \square \square \square \end{array} & \begin{array}{c} \square \square \square \end{array} \cdots \begin{array}{c} \square \square \square \square \end{array}
 \end{array}
 \end{aligned}$$

where $V_i^* = \text{Hom}_F(V_i, F)$ denotes the dual of V_i ; at each step of the resolution we have depicted the Young diagram of the corresponding partition. Our syzygy modules do not come equipped with an S_d -module structure, in particular one is free to drop the one-dimensional factors V_0^* and V_d^* if wanted, but these are included to emphasize the self-duality of the resolution. The space

$$V_1 = W_{\lambda_1} \cap L^{S_{\lambda_1}} = \text{span}\{\alpha_1, \alpha_2, \dots, \alpha_{d-1}\},$$

corresponding to the standard representation, seems missing, but will play a key role in the construction of the morphisms, and in fact the polynomial ring $R = F[x_1, \dots, x_{d-1}]$ will come about as $\text{Sym } V_1^*$ (as is also the case in Behnke's resolution from [2]).

(3.4). **Construction.** We now explain how the morphisms are constructed. We begin with the first step of the resolution (10). Start from the decomposition (8) of $\text{Sym}^2 V_{(d-1,1)}$ into irreducible subrepresentations. Let y_1, \dots, y_{d-1} be an F -basis for $V_{(d-1,1)}$ such that y_1 is fixed by S_{d-1} , and all other y_i are conjugate to y_1 in a way that is compatible with the action of S_d on L , i.e., for all $i = 1, \dots, d-1$ we have $\sigma_i(y) = y_i$. By Lemma 9 there is, up to scalar multiplication, a unique element in $V_{(d-2,2)}$ that is fixed by $S_2 \times S_{d-2}$. Under the isomorphism (8) this corresponds to an element

$$p^1 = \sum_{m,n=1}^{d-1} p_{mn}^1 y_m \otimes y_n \in \text{Sym}^2 V_{(d-1,1)}$$

for certain $p_{mn}^1 \in F$ where, without loss of generality, we may assume that $p_{mn}^1 = p_{nm}^1$ for all m, n . We use this element to construct a map

$$\psi_1 : \text{Sym}^2 V_1 \rightarrow V_2 : \alpha \otimes \beta \mapsto \sum_{m,n=1}^{d-1} p_{mn}^1 \alpha^{(m)} \beta^{(n)}$$

where as usual $\alpha^{(m)} = \sigma_m(\alpha)$ and $\beta^{(n)} = \sigma_n(\beta)$. Through dualization we obtain a map $\psi_1^* : V_2^* \rightarrow \text{Sym}^2 V_1^*$ whose codomain, after identifying V_1^* with

R_1 , can be viewed as R_2 . Thus this yields a map $V_2^* \otimes R(-2) \rightarrow V_0^* \otimes R$, as desired.

(3.5). Before discussing the next steps, we show that ψ_1 can be made quite concrete. The Specht module $V_{(d-2,2)}$ can be naturally realized inside $F[z_1, \dots, z_d]_2$. Explicitly, it is the subspace generated by $(z_1 - z_2)(z_3 - z_4)$ and all its conjugates [27, Prob. 4.47]. By Lemma 9, inside $F[z_1, \dots, z_d]_2 \cong V_{(d)}^2 \oplus V_{(d-1,1)}^2 \oplus V_{(d-2,2)}$, this subspace contains up to scalar multiplication a unique polynomial that is fixed by $S_2 \times S_{d-2}$. It is easily made explicit:

$$p^1 = \sum_{\tau \in S_{d-2}} (z_1 - z_{\tau(3)})(z_2 - z_{\tau(4)})$$

(abusively, we again call this polynomial p^1 ; here we recall from (2.1) that we view S_{d-2} as the subgroup of S_d fixing 1 and 2). This gives rise to a map

$$V_1 \rightarrow V_2 : \alpha \mapsto p^1(\alpha^{(1)}, \dots, \alpha^{(d)})$$

which is nothing but the quadratic map corresponding to the symmetric bilinear map ψ_1 from above; let us call it ψ'_1 . With respect to our bases $\alpha_1, \dots, \alpha_{d-1}$ of V_1 and $\omega_1^2, \dots, \omega_{\beta_1}^2$ of V_2 , it may be represented as

$$\psi'_1 \left(\sum_{i=j}^{d-1} a_j \alpha_j \right) = \sum_{\ell=1}^{\beta_1} \sum_{j,k=1}^{d-1} Q_{j,k}^\ell a_j a_k \omega_\ell^2,$$

for certain $Q_{j,k}^\ell \in F$, where we can assume $Q_{jk}^\ell = Q_{kj}^\ell$ for all j, k . Thus ψ_1 defines β_1 quadratic forms $Q^\ell \in R$. One checks that

$$\psi_1^* : V_2^* \rightarrow \text{Sym}^2 V_1^* : \omega_\ell^{2*} \mapsto \sum_{j,k=1}^{d-1} Q_{j,k}^\ell \alpha_j^* \otimes \alpha_k^*,$$

therefore the image of $V_2^* \otimes R(-2) \rightarrow R$ is the ideal generated by these Q^ℓ 's. We will soon prove that it equals I . We end by noting that, for $d = 4, 5$, the quadratic forms Q^ℓ can also be obtained by applying Bhargava's parametrizations from [3, 4] to the extension K/F .

(3.6). Now assume that $2 \leq i \leq d - 3$. Let y_1, \dots, y_{d-1} be as before and take an F -basis $w_1^i, \dots, w_{\beta_{i-1}}^i$ of V_{λ_i} such that w_1^i is fixed by S_{λ_i} , and all other w_j^i 's are conjugate to w_1^i . By Lemma 9 and Lemma 10 we have that

$$\text{mult}(V_{\lambda_{i+1}}, V_{\lambda_i} \otimes V_{(d-1,1)}) = 1, \quad \dim(V_{\lambda_{i+1}}^{S_{\lambda_{i+1}}}) = 1.$$

Hence there is, up to scalar multiplication, a unique element

$$p^i = \sum_{m=1}^{\beta_{i-1}} \sum_{n=1}^{d-1} p_{mn}^i w_m^i \otimes y_n \in V_{\lambda_i} \otimes V_{(d-1,1)}$$

which is fixed by $S_{\lambda_{i+1}}$ and such that its conjugates generate a representation isomorphic to $V_{\lambda_{i+1}}$. We use this p^i to construct a map

$$\psi_i : V_i \otimes V_1 \rightarrow V_{i+1} : \omega \otimes \alpha \mapsto \sum_{m=1}^{\beta_{i-1}} \sum_{n=1}^{d-1} p_{mn}^i \omega^{(m)} \alpha^{(n)},$$

where the conjugation of ω is labelled compatibly with that of w_1^i , that is, $\omega^{(m)} = \sigma(\omega)$ for any σ mapping w_1^i to w_m^i . By identifying V_1^* with R_1 as before, the dual map $\psi_i^* : V_{i+1}^* \rightarrow V_i^* \otimes V_1^*$ is naturally converted into a map $V_{i+1}^* \otimes R(-i-1) \rightarrow V_i^* \otimes R(-i)$.

(3.7). We have now constructed our resolution (10) except for the last step. For this, consider the representation $V_{\lambda_{d-2}} \otimes \text{Sym}^2 V_{(d-1,1)}$ and note that, by Corollary 12, it contains a unique subrepresentation isomorphic to $V_{(1,\dots,1)}$. As above, consider a basis $w_1^{d-2}, \dots, w_{\beta_{d-3}}^{d-2}$ of $V_{\lambda_{d-2}}$ such that w_1^{d-2} is fixed by $S_{\lambda_{d-2}}$ and the other w_j^{d-2} 's are conjugate to it. Then there exists an element

$$p^{d-2} = \sum_{i=1}^{\beta_{d-3}} \sum_{j,\ell=1}^{d-1} p_{ij\ell}^{d-2} w_i^{d-2} \otimes (y_j \otimes y_\ell),$$

unique up to scalar multiplication, on which S_d acts as the sign representation. We can assume that $p_{ij\ell}^{d-2} = p_{i\ell j}^{d-2}$, leading to a linear map

$$\psi_{d-2} : V_{d-2} \otimes \text{Sym}^2 V_1 \rightarrow V_d.$$

Upon dualizing, tensoring with R and identifying V_1^* with R_1 , this yields a morphism of graded R -modules $\psi_{d-2}^* : V_d^* \otimes R(-d) \rightarrow V_{d-2}^* \otimes R(-d+2)$.

(3.8). **A chain complex.** Our next goal is to prove that the sequence (10) is a complex. We first discuss what this means in terms of the maps ψ_i .

Lemma 14. *Assuming $d \geq 5$, the sequence (10) is a chain complex if and only if*

- (1) *the quadrics Q^ℓ vanish on the points (6),*
- (2) *we have $\psi_2(\psi_1(\alpha \otimes \alpha) \otimes \alpha) = 0$ for any $\alpha \in V_1$,*
- (3) *we have $\psi_{i+1}(\psi_i(\omega \otimes \alpha) \otimes \alpha) = 0$ for any $2 \leq i \leq d-4$, $\alpha \in V_1$ and $\omega \in V_i$,*
- (4) *we have $\psi_{d-2}(\psi_{d-3}(\omega \otimes \alpha) \otimes (\alpha \otimes \alpha)) = 0$ for any $\alpha \in V_1$ and $\omega \in V_{d-3}$.*

Proof. The first step $V_2^* \otimes R(-2) \rightarrow V_0^* \otimes R \rightarrow R/I$ being a complex is equivalent to the quadrics Q^ℓ being contained in the ideal I , i.e. the quadrics must vanish on (6). The subsequent steps are handled using a direct computation with bases.

For the second step of the resolution, we may write the maps ψ_1 and ψ_2 as

$$\psi_1(\alpha_j \otimes \alpha_k) = \sum_{\ell=1}^{\beta_1} Q_{jk}^\ell \omega_\ell^2, \quad \psi_2(\omega_\ell^2 \otimes \alpha_n) = \sum_{m=1}^{\beta_2} L_{\ell n}^m \omega_m^3,$$

for certain $Q_{jk}^\ell, L_{\ell n}^m$ in F . In terms of the dual bases with respect to $\text{Tr}_{L/F}$, the maps ψ_1^* and ψ_2^* satisfy

$$\psi_1^*(\omega_\ell^{2*}) = \sum_{j,k=1}^{d-1} Q_{jk}^\ell \alpha_j^* \otimes \alpha_k^*, \quad \psi_2^*(\omega_m^{3*}) = \sum_{\ell=1}^{\beta_1} \sum_{n=1}^{d-1} L_{\ell n}^m \omega_\ell^{2*} \otimes \alpha_n^*.$$

Let $q(x)$ be homogeneous in R , then in the sequence of maps a computation shows that

$$(11) \quad \psi_1^*(\psi_2^*(\omega_m^{3*} \otimes q(x))) = q(x) \sum_{\ell=1}^{\beta_1} \left(\sum_{n=1}^{d-1} L_{\ell n}^m x_n \right) \left(\sum_{j,k=1}^{d-1} Q_{jk}^\ell x_j x_k \right).$$

On the other hand, letting $\alpha = \sum_r a_r \alpha_r \in V_1$ for $a_r \in F$, one computes that

$$(12) \quad \psi_2(\psi_1(\alpha \otimes \alpha) \otimes \alpha) = \sum_{m=1}^{\beta_2} \left(\sum_{\ell=1}^{\beta_1} \left(\sum_{n=1}^{d-1} L_{\ell n}^m a_n \right) \left(\sum_{j,k=1}^{d-1} Q_{jk}^\ell a_j a_k \right) \right) \omega_m^3.$$

We indeed see that (12) is zero for all $\alpha \in V_1$ if and only if (11) is zero for all $m = 1, \dots, \beta_2$ and $q(x) \in R$. The “only if” part relies on the fact that a non-zero polynomial in R cannot vanish on all of F^{d-1} , because F is infinite: indeed, by assumption it admits an S_d -extension, so it cannot be finite.⁶

The middle steps are handled similarly. Explicitly, for $2 \leq i \leq d-4$ we may write the maps ψ_i and ψ_{i+1} as

$$\psi_i(\omega_j^i \otimes \alpha_k) = \sum_{\ell=1}^{\beta_i} L_{jk}^\ell \omega_\ell^{i+1}, \quad \psi_{i+1}(\omega_\ell^{i+1} \otimes \alpha_n) = \sum_{m=1}^{\beta_{i+1}} L_{\ell n}^m \omega_m^{i+2},$$

for certain $L_{jk}^\ell, L_{\ell n}^m \in F$. On the dual bases this gives

$$\psi_i^*(\omega_\ell^{i+1*}) = \sum_{j=1}^{\beta_{i-1}} \sum_{k=1}^{d-1} L_{jk}^\ell \omega_j^{i*} \otimes \alpha_k^*, \quad \psi_{i+1}^*(\omega_m^{i+2*}) = \sum_{\ell=1}^{\beta_i} \sum_{n=1}^{d-1} L_{\ell n}^m \omega_\ell^{i+1*} \otimes \alpha_n^*.$$

For $q(x) \in R$, one computes that

$$(13) \quad \psi_i^*(\psi_{i+1}^*(\omega_m^{i+2*} \otimes q(x))) = \sum_{j=1}^{\beta_{i-1}} \omega_j^{i*} \otimes \left(q(x) \sum_{\ell=1}^{\beta_i} \left(\sum_{n=1}^{d-1} L_{\ell n}^m x_n \right) \left(\sum_{k=1}^{d-1} L_{jk}^\ell x_k \right) \right).$$

⁶But even for a finite field F it is true that a non-zero polynomial of degree $< \text{char } F$ cannot vanish everywhere.

On the other hand, if we let $\alpha = \sum_r a_r \alpha_r$ for $a_r \in F$, then

$$(14) \quad \psi_{i+1}(\psi_i(\omega_j^i \otimes \alpha) \otimes \alpha) = \sum_{m=1}^{\beta_{i+1}} \left(\sum_{\ell=1}^{\beta_i} \left(\sum_{n=1}^{d-1} L_{\ell n}^m a_n \right) \left(\sum_{k=1}^{d-1} L_{jk}^\ell a_k \right) \right) \omega_m^{i+2}.$$

Again, it is clear that (14) is zero for all $j = 1, \dots, \beta_{i-1}$ and all $\alpha \in V_1$ if and only if (13) is zero for all $m = 1, \dots, \beta_{i+1}$ and all $q(x) \in R$.

We omit the details of the final step, which again can be dealt with analogously. \square

The reader may have observed that the syzygies of our resolution make an explicit appearance in the above proof, more precisely in (11) and (13).

(3.9). So we turn to proving the statements (1–4) from Lemma 14. Towards proving that the quadrics vanish as wanted, we give an explicit description of the coordinates of our points (6); this again follows [4, §2]. Consider the matrix

$$(15) \quad D = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1^{(1)} & \alpha_1^{(2)} & \dots & \alpha_1^{(d)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{d-1}^{(1)} & \alpha_{d-1}^{(2)} & \dots & \alpha_{d-1}^{(d)} \end{pmatrix},$$

and denote by $D_{j,i}$ the minor of D corresponding to $\alpha_j^{(i)}$, i.e. it is $(-1)^{i+j}$ times the determinant of D with the j -th row and i -th column removed. Then we have $\alpha_j^{*(i)} = D_{j+1,i} / \det D$.

Theorem 15. *The quadrics Q^ℓ vanish on the points (6), i.e. for any ℓ we have*

$$\sum_{i,j=1}^{d-1} Q_{ij}^\ell \alpha_i^* \alpha_j^* = 0.$$

Proof. We claim that for $p = 1, \dots, d$ we have

$$(16) \quad \sum_{m,n=0}^{d-1} \psi_1(\alpha_m^* \otimes \alpha_n^*) \alpha_m^{(p)} \alpha_n^{(p)} = 0.$$

Indeed, using the definition of ψ_1 we can expand this as

$$\sum_{\substack{i,j=1 \\ i \neq j}}^d c_{ij} \sum_{m,n=0}^{d-1} D_{m+1,i} D_{n+1,j} \alpha_m^{(p)} \alpha_n^{(p)}$$

for certain coefficients $c_{ij} \in F$ in which we have absorbed a denominator $\det D^2$ (which is a non-zero element of F , being the discriminant of $1, \alpha_1, \dots, \alpha_{d-1}$ with respect to K/F). In each term, at least one of i, j is

not equal to p . Say $i \neq p$, then the corresponding summand can be written as

$$c_{ij} \cdot \sum_{n=1}^{d-1} D_{n+1,j} \alpha_n^{(p)} \cdot \sum_{m=1}^{d-1} D_{m+1,i} \alpha_m^{(p)} = 0,$$

and one sees that the last factor is 0 because it is the determinant of the matrix obtained from D by replacing the i th column with a copy of the p th column. The case $j \neq p$ is analogous, hence the claim follows.

Expand $\alpha_m \alpha_n$ with respect to the basis $1, \alpha_1, \dots, \alpha_{d-1}$ and let $c_{mn}^q \in F$ denote the coordinate at α_q . Then we may rewrite (16) as

$$\sum_{m,n=0}^{d-1} \sum_{q=0}^{d-1} \psi_1(\alpha_m^* \otimes \alpha_n^*) c_{mn}^q \alpha_q^{(p)} = 0.$$

Multiplying by $\alpha_r^{*(p)}$ and summing over p yields

$$\sum_{m,n=0}^{d-1} c_{mn}^r \psi_1(\alpha_m^* \otimes \alpha_n^*) = 0$$

for all $r = 0, \dots, d-1$. Since $\alpha_m^* = \sum_{i=1}^{d-1} \text{Tr}_{L/F}(\alpha_i^* \alpha_m^*) \alpha_i$ (here we use the fact that the α_i have trace zero), we get for any ℓ

$$\sum_{m,n=0}^{d-1} c_{mn}^r \sum_{i,j=1}^{d-1} Q_{ij}^\ell \text{Tr}_{L/F}(\alpha_i^* \alpha_m^*) \text{Tr}_{L/F}(\alpha_j^* \alpha_n^*) = 0.$$

Multiplying by α_r and summing over r finally yields the desired vanishing. \square

(3.10). As for the other statements in Lemma 14, we have:

Lemma 16. *If $d \geq 5$, then for $\alpha \in V_1$, we have that $\psi_2(\psi_1(\alpha \otimes \alpha) \otimes \alpha) = 0$.*

Proof. Let $\eta = \psi_2(\psi_1(\alpha \otimes \alpha) \otimes \alpha)$. By construction of ψ_2 , η lies in W_{λ_3} . On the other hand, η is also an element of the subrepresentation $U = \text{span}\{\alpha^{(j)} \alpha^{(k)} \alpha^{(\ell)} \mid 1 \leq j, k, \ell \leq d\}$ of L . Now U is the image of the equivariant map

$$\text{Sym}^3 V_{(d-1,1)} \rightarrow L : y_j \otimes y_k \otimes y_\ell \mapsto \alpha^{(j)} \alpha^{(k)} \alpha^{(\ell)}.$$

However, by Lemma 13, $\text{Sym}^3 V_{(d-1,1)}$ does not contain the irreducible representation V_{λ_3} and so indeed $\eta = 0$. \square

Lemma 17. *For $2 \leq i \leq d-4$, $\alpha \in V_1$ and $\omega \in V_i$, we have $\psi_{i+1}(\psi_i(\omega \otimes \alpha) \otimes \alpha) = 0$.*

Proof. This is similar to the previous lemma. Let $\eta = \psi_{i+1}(\psi_i(\omega \otimes \alpha) \otimes \alpha)$. Then η lives in $W_{\lambda_{i+2}}$, by construction of ψ_{i+1} . However, η is also contained in the subrepresentation $U = \text{span}\{\sigma(\omega) \alpha^{(j)} \alpha^{(k)} \mid \sigma \in S_d, 1 \leq j, k \leq d\}$. The space U is the image of the map

$$V_{\lambda_i} \otimes \text{Sym}^2 V_{(d-1,1)} \rightarrow L : \sigma(w_1^i) \otimes (y_j \otimes y_k) \mapsto \sigma(\omega) \alpha^{(j)} \alpha^{(k)}.$$

However, by Corollary 12, $V_{\lambda_i} \otimes \text{Sym}^2 V_{(d-1,1)}$ does not contain the representation $V_{\lambda_{i+2}}$, proving the desired result. \square

Lemma 18. *If $d \geq 5$ then we have $\psi_{d-2}(\psi_{d-3}(\omega \otimes \alpha) \otimes (\alpha \otimes \alpha)) = 0$ for all $\alpha \in V_1$ and $\omega \in V_{d-3}$.*

Proof. The proof idea is the same as in the previous two lemmas. This time one uses that $V_{\lambda_{d-3}} \otimes \text{Sym}^3 V_{(d-1,1)}$ does not contain the sign representation $V_{(1,\dots,1)}$, by Lemma 13. \square

(3.11). The case $d = 4$ is not covered by our treatment so far. We leave it to the reader to check that the requirements from Lemma 14 should be replaced by

- (1) the quadrics Q^ℓ vanish on the points (6),
- (2) we have $\psi_2(\psi_1(\alpha \otimes \alpha) \otimes (\alpha \otimes \alpha)) = 0$ for any $\alpha \in V_1$.

The first requirement is covered by Theorem 15, while the second requirement can be checked as in Lemma 18, with $\text{mult}(V_{(1^4)}, \text{Sym}^4 V_{(3,1)}) = 0$ from (9) as the key representation-theoretic ingredient.

(3.12). **Exactness and minimality.** We now prove that our chain complex is indeed a minimal free resolution. This is equivalent with proving surjectivity of the maps ψ_i :

Lemma 19. *If the linear maps $\psi_1 : \text{Sym}^2 V_1 \rightarrow V_2$, $\psi_i : V_i \otimes V_1 \rightarrow V_{i+1}$ for $2 \leq i \leq d-3$, and $\psi_{d-2} : V_{d-2} \otimes \text{Sym}^2 V_1 \rightarrow V_d$ are all surjective, then the complex (10) is a minimal free resolution.*

Proof. From [45, (4.2)] we know that I is generated by β_1 linearly independent quadrics. Thanks to Theorem 15 we also know that the β_1 quadrics Q^ℓ are elements of I . Thus they form a minimal generating set if and only if they are linearly independent. But this is equivalent to ψ_1 being surjective. Assuming surjectivity of ψ_1 , again from [45, (4.2)] we then know that the R -module of syzygies between these Q^ℓ 's is generated by β_2 linearly independent linear syzygies. But we have just showed that ψ_2^* produces β_2 linear syzygies. Thus they form a minimal generating set if and only if they are linearly independent. In turn, this is equivalent to the surjectivity of ψ_2 . An inductive application of this argument concludes the proof. \square

(3.13). To prove surjectivity we need the following technical lemma; here, by the product $W \cdot W'$ of $W, W' \subseteq L$ we mean the subspace generated by all $w w'$ for $w \in W$ and $w' \in W'$.

Lemma 20. *We have $W_{\lambda_2} \subseteq W_{(d-1,1)}^2$, $W_{\lambda_{i+1}} \subseteq W_{\lambda_i} \cdot W_{(d-1,1)}$ for $2 \leq i \leq d-3$, and $W_{(1^d)} \subseteq W_{\lambda_{d-2}} \cdot W_{(d-1,1)}^2$.*

Proof. Denoting by $\text{Sym}^2 W_{(d-1,1)}$ the subspace of L generated by all elements of the form $\alpha^{(m)} \beta^{(n)} + \alpha^{(n)} \beta^{(m)}$ with $\alpha, \beta \in V_1$ and $m, n \in \{1, \dots, d\}$, we will first show that

$$(17) \quad W_{\lambda_{i+1}} \subseteq W_{(d-1,1)}^{i-1} \text{Sym}^2 W_{(d-1,1)}$$

for all $i = 1, \dots, d-3$. Since $\text{Sym}^2 W_{(d-1,1)} \subseteq W_{(d-1,1)}^2$, this will settle the first inclusion.

Because of the S_d -action, it is enough to prove that V_{i+1} is contained in the right-hand side of (17). In turn, it suffices to prove this for $L^{S_{\lambda_{i+1}}} \supseteq V_{i+1}$. Since this field is generated by $\alpha^{(d-i)} + \alpha^{(d-i+1)}$, $\alpha^{(d-i+2)}$, $\alpha^{(d-i+3)}$, \dots , $\alpha^{(d)}$ for some primitive element $\alpha \in K$, any element is an F -linear combination of elements of the form

$$(\alpha^{(d-i)} + \alpha^{(d-i+1)})^{f_1} \alpha^{(d-i+2)f_2} \dots \alpha^{(d)f_i}.$$

Every appearance of $\alpha^{(m)f_j}$ can be rewritten as an F -linear combination of the elements $\alpha_0^{(m)} = 1, \alpha_1^{(m)}, \dots, \alpha_{d-1}^{(m)}$. After doing this, we find that every element of $L^{S_{\lambda_{i+1}}}$ is an F -linear combination of elements of the form

$$\left(\alpha_{k_0}^{(d-i)} \alpha_{k_1}^{(d-i+1)} + \alpha_{k_0}^{(d-i+1)} \alpha_{k_1}^{(d-i)} \right) \alpha_{k_2}^{(d-i+2)} \dots \alpha_{k_i}^{(d)},$$

for $0 \leq k_j \leq d-1$. This shows that V_{i+1} is contained in

$$(18) \quad (W_{(d)} + W_{(d-1,1)})^{i-1} \text{Sym}^2(W_{(d)} + W_{(d-1,1)}).$$

We claim that $\text{Sym}^2 W_{(d-1,1)}$ contains $W_{(d)}$ and $W_{(d-1,1)}$. This claim readily implies that $\text{Sym}^2(W_{(d)} + W_{(d-1,1)}) = \text{Sym}^2 W_{(d-1,1)}$ and also that

$$(W_{(d)} + W_{(d-1,1)}) \text{Sym}^2 W_{(d-1,1)} = W_{(d-1,1)} \text{Sym}^2 W_{(d-1,1)},$$

so that (18) equals $W_{(d-1,1)}^{i-1} \text{Sym}^2 W_{(d-1,1)}$, thus settling (17).

In order to prove the claim, it suffices to show that $\text{Sym}^2 W_{(d-1,1)}$ contains $K = V_0 + V_1$. Pick a non-zero element $\beta_1 \in K \cap \text{Sym}^2 W_{(d-1,1)}$ that is not contained in F ; e.g. one of $\alpha_1^2, \alpha_1 \alpha_2$ will do. By replacing it with

$$\beta_1 - \frac{1}{d!} \text{Tr}_{L/F}(\beta_1) = \beta_1 - \frac{1}{d} \sum_{j=1}^d \beta_1^{(j)}$$

if needed, we can assume that it has trace zero, so that it belongs to V_1 . Extending this to a basis $\beta_1, \dots, \beta_{d-1}$ of V_1 , it is easy to see that $\beta_1^2, \beta_1 \beta_2, \dots, \beta_1 \beta_{d-1}, \beta_1 \in K \cap \text{Sym}^2 W_{(d-1,1)}$ are linearly independent over F , hence they must generate K , as wanted.

Having settled the case $i = 1$, we now assume $2 \leq i \leq d-3$. Decomposing

$$V_{(d-1,1)}^{\otimes i-2} \otimes \text{Sym}^2 V_{(d-1,1)} = \bigoplus_{\lambda} V_{\lambda}^{c_{i,\lambda}}$$

we see from (17) that

$$W_{\lambda_{i+1}} \subseteq W_{(d-1,1)}^{i-1} \text{Sym}^2 W_{(d-1,1)} \subseteq \sum_{\substack{\lambda \text{ with} \\ c_{i,\lambda} > 0}} W_{\lambda} \cdot W_{(d-1,1)}.$$

By decomposing $\text{Sym}^2 V_{(d-1,1)}$ as in (8) and using Lemma 10 and Lemma 11, one sees that $c_{i,\lambda} > 0$ only if the Young diagram corresponding to λ has at most i boxes not in its first row and $\lambda \neq (d-i, 1^i)$. Consequently, again

using Lemma 10, we see that the only partition λ with $c_{i,\lambda} > 0$ such that $V_\lambda \otimes V_{(d-1,1)}$ contains $V_{\lambda_{i+1}}$ is $\lambda = \lambda_i$. We conclude that

$$W_{\lambda_{i+1}} \subseteq W_{\lambda_i} \cdot W_{(d-1,1)},$$

as wanted.

For the last inclusion we give an ad-hoc proof. Recall that $W_{(1^d)}$ is generated by the Vandermonde determinant

$$\begin{vmatrix} 1 & \alpha^{(1)} & \dots & \alpha^{(1)d-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{(d)} & \dots & \alpha^{(d)d-1} \end{vmatrix}$$

for some primitive element $\alpha \in K$. Expanding this determinant with respect to the last two columns yields a linear combination of

$$\begin{vmatrix} 1 & \alpha^{(1)} & \dots & \alpha^{(1)d-3} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{(d-2)} & \dots & \alpha^{(d-2)d-3} \end{vmatrix} \cdot (\alpha^{(d)} - \alpha^{(d-1)}) \cdot (\alpha^{(d-1)} \alpha^{(d)})^{d-1}$$

and conjugates thereof. By [27, Prob. 4.47] the product of the first two factors is in the image of the Specht module $V_{\lambda_{d-2}}$ under the natural map $F[z_1, \dots, z_d] \rightarrow L : z_i \mapsto \alpha^{(i)}$, so it is in $W_{\lambda_{d-2}}$. By mimicking the argumentation from the start of this proof, the last factor is seen to be in

$$(W_{(d)} + W_{(d-1,1)})^2 = W_{(d-1,1)}^2,$$

where the equality holds because $W_{(d-1,1)}^2 \supseteq \text{Sym}^2 W_{(d-1,1)}$ contains $W_{(d)}$ and $W_{(d-1,1)}$. \square

(3.14). We are now ready to conclude:

Lemma 21. *The maps ψ_i are all surjective, so that the complex (10) is a minimal free resolution.*

Proof. We first prove the surjectivity of $\psi_1 : \text{Sym}^2 V_1 \rightarrow V_2$. By Lemma 20 any $\gamma \in V_2$ can be written as

$$\gamma = \sum_{\substack{1 \leq m, n \leq d \\ 1 \leq j \leq k \leq d-1}} c_{jk}^{mn} \alpha_j^{(m)} \alpha_k^{(n)}$$

for $c_{jk}^{mn} \in F$. For every pair j, k define $D_{jk} = \sum_{1 \leq m, n \leq d} c_{jk}^{mn} \alpha_j^{(m)} \alpha_k^{(n)}$. We show that we may take the individual D_{jk} 's to be in V_2 as well. For this, let $\rho \in Z(F[S_d])$ be the element obtained by applying Lemma 8 to the partition $(d-2, 2)$. Now consider for every j, k the element

$$D'_{jk} = \frac{1}{2(d-2)!} \sum_{\tau \in S_2 \times S_{d-2}} \tau(\rho(D_{jk})).$$

We still have $\gamma = \sum_{1 \leq j \leq k \leq d-1} D'_{jk}$ with $D'_{jk} \in \text{span}\{\alpha_j^{(m)} \alpha_k^{(n)} \mid 1 \leq m, n \leq d\}$, and each D'_{jk} is contained in $L^{S_2 \times S_{d-2}} \cap W_{(d-2,2)} = V_2$, as wanted.

We now prove that every D'_{jk} is in the image of ψ_1 . If $D'_{jk} = 0$, there is nothing to prove. So we assume that $D'_{jk} \neq 0$. Consider the equivariant map

$$\phi_{jk} : V_{(d-1,1)} \otimes V_{(d-1,1)} \rightarrow L : y_m \otimes y_n \mapsto \alpha_j^{(m)} \otimes \alpha_k^{(n)}$$

whose image is precisely $\text{span}\{\alpha_j^{(m)} \alpha_k^{(n)} \mid 1 \leq m, n \leq d\}$. From Lemma 10 we see that $V_{(d-1,1)} \otimes V_{(d-1,1)}$ contains a unique copy of $V_{(d-2,2)}$, which contains the element p^1 from (3.4). The image of ϕ_{jk} also contains $V_{(d-2,2)}$ as a subrepresentation because $D'_{jk} \neq 0$. So Schur's lemma implies that ϕ_{jk} cannot map p^1 to zero. Since both $\phi_{jk}(p^1)$ and D'_{jk} are fixed under the action of $S_2 \times S_{d-2}$, we find from Lemma 9 that $D'_{jk} = d_{jk} \phi_{jk}(p^1) = \psi_1(d_{jk} \alpha_j \otimes \alpha_k)$ for some $d_{jk} \in F^\times$. But then

$$\gamma = \psi_1 \left(\sum_{1 \leq i \leq j \leq d-1} d_{jk} \alpha_j \otimes \alpha_k \right),$$

i.e. ψ_1 is surjective.

For the other maps, the argument is similar. Let $2 \leq i \leq d-3$ and take $\gamma \in V_{i+1}$. We wish to prove that γ is in the image of $\psi_i : V_i \otimes V_1 \rightarrow V_{i+1}$. By Lemma 20 we may write γ as

$$\gamma = \sum_{\substack{1 \leq j \leq \beta_{i-1} \\ 1 \leq k \leq d-1 \\ \sigma \in S_d, 1 \leq n \leq d}} c_{jk}^{\sigma, n} \sigma(\omega_j^i) \alpha_k^{(n)},$$

for $c_{jk}^{\sigma, n} \in F$. For every pair j, k define

$$D_{jk} = \sum_{\substack{\sigma \in S_d \\ 1 \leq n \leq d}} c_{jk}^{\sigma, n} \sigma(\omega_j^i) \alpha_k^{(n)}.$$

Let $\rho \in Z(F[S_d])$ come from Lemma 8 applied to λ_{i+1} and define

$$D'_{jk} = \frac{1}{2(d-i-1)!} \sum_{\tau \in S_{\lambda_{i+1}}} \tau(\rho(D_{jk})).$$

We still have $\gamma = \sum_{1 \leq j \leq \beta_{i-1}, 1 \leq k \leq d-1} D'_{jk}$ with $D'_{jk} \in \text{span}\{\sigma(\omega_j^i) \alpha_k^{(n)} \mid \sigma \in S_d, 1 \leq n \leq d\}$, and each D'_{jk} is contained in $L^{S_{\lambda_{i+1}}} \cap W_{\lambda_{i+1}} = V_{i+1}$. We prove that every D'_{jk} is in the image of ψ_i , from which the surjectivity of ψ_i follows. We may assume that $D'_{jk} \neq 0$, for else there is nothing to prove. Consider the equivariant map

$$\phi_{jk} : V_{\lambda_i} \otimes V_{(d-1,1)} \rightarrow L : \sigma(\omega_j^i) \otimes y_n \mapsto \sigma(\omega_j^i) \alpha_k^{(n)}.$$

The image of this map is precisely $\text{span}\{\sigma(\omega_j^i) \alpha_k^{(n)} \mid \sigma \in S_d, 1 \leq n \leq d\}$. Lemma 10, Schur's lemma and the fact that $D'_{jk} \neq 0$ yields as before that $D'_{jk} = d_{jk} \phi_{jk}(p^i) = \psi_i(d_{jk} \omega_j^i \otimes \alpha_k)$ for some $d_{jk} \in F^\times$, as wanted.

The surjectivity of the last map ψ_{d-2} can be proved similarly, the main representation-theoretic ingredient now being that $V_{\lambda_{d-2}} \otimes \text{Sym}^2 V_{(d-1,1)}$ contains a unique component $V_{(1^d)}$ by Corollary 12. We omit further details. \square

(3.15). **Three points in \mathbf{P}^1 .** The remainder of Section 3 discusses two stand-alone observations, which can be skipped by the reader eager to move forward. First, we consider the ideal I of the 3 points in \mathbf{P}^1 associated with a basis $1, \alpha_1, \alpha_2$ of a cubic S_3 -extension K/F as in (1.8), where we assume $\text{Tr}_{L/F}(\alpha_i) = 0$ for $i = 1, 2$. This ideal is generated by one cubic polynomial in R rather than by quadrics, hence the resolution takes the form

$$0 \rightarrow R(-3) \rightarrow R \rightarrow R/I \rightarrow 0.$$

This exceptional behaviour also has a representation-theoretic explanation: for $d = 3$ the decomposition (8) fails, instead we have $\text{Sym}^2 V_{(2,1)} \cong V_{(3)} \oplus V_{(2,1)}$, so we cannot construct the quadratic map from (3.4). Here, the correct representation to look at is $\text{Sym}^3 V_{(2,1)}$, which has a unique component $V_{(1^3)}$ in view of (9). Let

$$p = \sum_{j,k,\ell=1}^2 p_{jkl} y_j \otimes y_k \otimes y_\ell \in \text{Sym}^3 V_{(2,1)}$$

be a generator of this component, expanded with respect to some basis y_1, y_2 of $V_{(2,1)}$, this defines the cubic map

$$\psi : V_1 \rightarrow V_3 : \alpha \mapsto p(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)})$$

from $V_1 = W_{(2,1)} \cap L^{S_2}$ to $V_3 = W_{(1,1,1)}$, which is easy to make explicit:

$$\psi(\alpha) = (\alpha^{(1)} - \alpha^{(2)})(\alpha^{(1)} - \alpha^{(3)})(\alpha^{(2)} - \alpha^{(3)}).$$

This naturally induces a linear map $\text{Sym}^3 V_1 \rightarrow V_3$, mapping $\alpha_j \otimes \alpha_k \otimes \alpha_\ell$ to $C_{jkl} \delta$, for some $C_{jkl} \in F$ and some fixed non-zero $\delta \in V_3$, corresponding to a cubic form

$$C(x_1, x_2) = \sum_{j,k,\ell=1}^2 C_{jkl} x_j x_k x_\ell$$

which again can be seen to vanish on the points (6). The resulting complex $0 \rightarrow V_3^* \otimes R(-3) \rightarrow V_0^* \otimes R \rightarrow R/I \rightarrow 0$ is a minimal free resolution of R/I . We note that the cubic form C can also be obtained by applying the Delone–Faddeev parametrization [28, Prop. 4.2] to the cubic extension K/F .

(3.16). **Self-duality.** As a second side trip, we note that there is a “dual” representation-theoretic construction of a minimal free resolution of our coordinate ring R/I , in which the spaces $V_0^*, V_2^*, V_3^*, \dots, V_{d-2}^*, V_d^*$ appear in opposite order. This construction links the well-known self-duality for minimal free resolutions of d general points in \mathbf{P}^{d-2} [2, 45] to the duality for representations of S_d , obtained by tensoring with the sign representation $V_{\lambda_d} = V_{(1^d)}$ (i.e. by transposing Young diagrams).

(3.17). For the first step, tensoring the decomposition of $V_{(d-2,2)}$ from (8) with the sign representation yields

$$V_{(1^d)} \otimes \text{Sym}^2 V_{(d-1,1)} \cong V_{(1^d)} \oplus V_{(2,1^{d-2})} \oplus V_{(2^2,1^{d-4})},$$

so there is a unique component $V_{\lambda_{d-2}}$. Moreover, Lemma 9 shows that $V_{\lambda_{d-2}}^{S_{\lambda_{d-2}}}$ has dimension 1. Denote by u a generator of $V_{(1,\dots,1)}$. Let

$$q^1 = \sum_{m,n=1}^{d-1} q_{mn}^1 u \otimes (y_m \otimes y_n) \in V_{(1,\dots,1)} \otimes \text{Sym}^2 V_{(d-1,1)}$$

be the unique element (up to multiplication by a scalar from F^\times) which is fixed under $S_{\lambda_{d-2}}$ and such that its conjugates generate the representation $V_{\lambda_{d-2}}$. We can assume that $q_{mn}^1 = q_{nm}^1$ and use this element to define a quadratic map

$$\varphi_1 : V_d \otimes V_1 \rightarrow V_{d-2} : \delta \otimes \alpha \mapsto q^1(\delta, \alpha^{(1)}, \dots, \alpha^{(d)}),$$

which naturally induces a linear map $V_d \otimes \text{Sym}^2 V_1 \rightarrow V_{d-2}$, also denoted by φ_1 . Dualizing yields the first step of the resolution for I

$$V_{d-2}^* \otimes R(-2) \xrightarrow{\varphi_1^*} V_d^* \otimes R \rightarrow R/I \rightarrow 0.$$

The subsequent steps are similar. Let $2 \leq i \leq d-3$, then from Lemma 10 we see that $V_{\lambda_{d-i-1}}$ appears once in $V_{\lambda_{d-i}} \otimes V_{(d-1,1)}$. By Lemma 9 there is, up to scalar multiplication, a unique

$$q^i = \sum_{m=1}^{\beta_{d-i-1}} \sum_{n=1}^{d-1} q_{mn}^i w_m^{d-i} \otimes y_n \in V_{\lambda_{d-i}} \otimes V_{(d-1,1)},$$

that is fixed by $S_{\lambda_{d-i-1}}$ and whose conjugates generate a $V_{\lambda_{d-i-1}}$.⁷ In the same way as before, this induces a linear map

$$\varphi_i : V_{d-i} \otimes V_1 \rightarrow V_{d-i-1}$$

which, when dualized, yields a next step in the resolution.

To conclude, using Lemma 10 and Lemma 11 along with (8) one finds a unique trivial component in the representation $V_{(d-2,2)} \otimes \text{Sym}^2 V_{(d-1,1)}$, resulting in a linear map $\varphi_{d-2} : V_2 \otimes \text{Sym}^2 V_1 \rightarrow V_0$ whose dual is the last step of the resolution.

⁷Notice that for $3 \leq i \leq d-3$ the elements p^{d-i} from (3.6) are also taken from $V_{\lambda_{d-i}} \otimes V_{(d-1,1)}$, but they are fixed under $S_{\lambda_{d-i+1}}$ and generate a $V_{\lambda_{d-i+1}}$, rather than being fixed under $S_{\lambda_{d-i-1}}$ and generating a $V_{\lambda_{d-i-1}}$.

(3.18). The resulting sequence

$$\begin{aligned} 0 \rightarrow V_0^* \otimes R(-d) \xrightarrow{\varphi_{d-2}^*} V_2^* \otimes R(-d+2) \xrightarrow{\varphi_{d-3}^*} \dots \\ \dots \xrightarrow{\varphi_3^*} V_{d-3}^* \otimes R(-3) \xrightarrow{\varphi_2^*} V_{d-2}^* \otimes R(-2) \xrightarrow{\varphi_1^*} V_d^* \otimes R \rightarrow R/I \rightarrow 0 \end{aligned}$$

is indeed a minimal graded free resolution because it is explicitly isomorphic to our sequence from (3.3). We sketch this for steps $2 \leq i \leq d-3$; the analysis for $i=1$ and $i=d-2$ is completely similar. Choose elements $c_\sigma^i \in F$, where σ ranges over S_d , with the following property: if $w \in V_{\lambda_i}$ is fixed by S_{λ_i} , then $\sum_{\sigma \in S_d} c_\sigma^i \sigma(u \otimes w)$ is non-zero and fixed under $S_{\lambda_{d-i}}$. Note that such coefficients indeed exist in view of Lemma 9, because the $\sigma(u \otimes w)$'s span a $V_{(1, \dots, 1)} \otimes V_{\lambda_i} \cong V_{\lambda_{d-i}}$. It follows, for each $m = 1, \dots, \beta_{i-1} = \beta_{d-i-1}$, that $\sum_{\sigma \in S_d} c_\sigma^i \sigma(\delta \omega_m^i) \in W_{\lambda_{d-i}}$ is fixed under $S_{\lambda_{d-i}}$ and non-zero; consequently

$$\varpi_m^i = \sum_{\sigma \in S_d} c_\sigma^i \sigma(\delta \omega_m^i), \quad m = 1, \dots, \beta_{d-i-1}$$

is a basis for V_{d-i} . We leave it to the reader to verify that the map ψ_i expressed with respect to the bases ω_m^i and ω_m^{i+1} coincides with the map φ_i expressed with respect to the bases ϖ_m^i and ϖ_m^{i+1} .

4. SCROLLAR INVARIANTS OF REPRESENTATIONS AND RESOLVENTS

(4.1). We now return to our simply branched degree d cover $\varphi : C \rightarrow \mathbf{P}^1$ over a field k satisfying $\text{char } k = 0$ or $\text{char } k > d$, with C a curve of genus g . In this section we explain the decomposition (3) that underlies Definition 1, introducing the scrollar invariants of C with respect to a partition $\lambda \vdash d$: this is done in (4.5). We then proceed with studying some of their first properties. Recall that this notion generalizes that of the scrollar invariants e_1, e_2, \dots, e_{d-1} of C with respect to φ , which arise as the scrollar invariants with respect to the partition $(d-1, 1)$. Several other examples are discussed, which are all subsumed by Proposition 2 on the scrollar invariants of hooks: its proof can be found in (4.11). In (4.9) we prove our ‘‘volume formula’’ generalizing the sum formula $e_1 + e_2 + \dots + e_{d-1} = g + d - 1$, and we also present an explicit formula relating the scrollar invariants of λ to those of the dual partition λ^* . The main result of this section is Theorem 4, expressing the scrollar invariants of resolvent covers in terms of scrollar invariants with respect to certain partitions: a proof can be found in (4.6). Our treatment is based on explicit function field arithmetic, in the style of Hess [30]; alternatively, it should be possible to develop much of the following material using the formalism of parabolic vector bundles and the Mehta–Seshadri correspondence [38] mentioned in (1.4).

(4.2). **Reduced bases.** Consider the extension $k(t) = k(\mathbf{P}^1) \subseteq k(C) = K$ induced by φ . The simple branching assumption will not play an important role until (4.8). Following Hess, see [30] and [31, pp. 43-52], we can interpret the decomposition (1) in terms of this extension. Denote by \mathcal{O}_K resp. \mathcal{O}_K^∞

the integral closure of $k[t]$ resp. $k[t^{-1}]$ in K . Geometrically, these rings correspond to two affine patches of the curve C , one above $\mathbf{A}^1 = \mathbf{P}^1 \setminus \{\infty\}$ and the other above $\mathbf{P}^1 \setminus \{0\}$.

Theorem 22. *There exists a $k[t]$ -basis $1, \alpha_1, \dots, \alpha_{d-1}$ of \mathcal{O}_K , together with unique integers $1 \leq r_1 \leq \dots \leq r_{d-1}$ such that $1, t^{-r_1}\alpha_1, \dots, t^{-r_{d-1}}\alpha_{d-1}$ is a $k[t^{-1}]$ -basis of \mathcal{O}_K^∞ .*

Proof. This follows from [30, Cor. 4.3]; see also [31, p. 46]. \square

The integers r_i describe how $\varphi_*\mathcal{O}_C$ splits, that is, $r_i = e_i$ for $i = 1, \dots, d-1$. An accompanying basis $1, \alpha_1, \dots, \alpha_{d-1}$ as above is called a “reduced basis”.

(4.3). As explained in [30, §7] and already touched upon in (1.1), the notion of a reduced basis is the function-field theoretic analogue of that of a Minkowski-reduced \mathbf{Z} -basis $1, v_1, \dots, v_{d-1}$ of the ring of integers of a degree d number field E , and under this correspondence the scrollar invariants e_i can be seen as the equivalents of $\log \|v_i\|$, with $\|\cdot\|$ denoting the ℓ_2 -norm under the Minkowski embedding. Given the analogy between g and $\log \sqrt{|\Delta_E|}$, where Δ_E is the discriminant of E , it is interesting to view the property $e_1 + e_2 + \dots + e_{d-1} = g + d - 1$ against Minkowski’s second theorem

$$\|v_1\| \cdot \|v_2\| \cdots \|v_{d-1}\| \sim_d \sqrt{|\Delta_E|},$$

and to view the Maroni bound $e_{d-1} \leq (2g + 2d - 2)/d$ against the observation $\|v_{d-1}\| = O_d(|\Delta_E|^{1/d})$ due to Peikert–Rosen [41, Lem. 5.4], see also [6, Thm. 1.6].

(4.4). For the purposes of this paper, it is convenient to assume that $\text{Gal}(L/k(t))$ is the full symmetric group S_d , although the observations below apply more generally. Let \mathcal{O}_L resp. \mathcal{O}_L^∞ be the integral closure of $k[t]$ resp. $k[t^{-1}]$ in L . For any $k(t)$ -vector space $V \subseteq L$, say of dimension n , we define

$$\mathcal{O}_V = \mathcal{O}_L \cap V, \quad \mathcal{O}_V^\infty = \mathcal{O}_L^\infty \cap V.$$

Note that \mathcal{O}_V is a free $k[t]$ -submodule of L of rank n , and similarly for \mathcal{O}_V^∞ . By [30] there are unique integers $0 \leq r_1 \leq \dots \leq r_n$, together with a $k[t]$ -basis v_1, \dots, v_n of \mathcal{O}_V such that

$$t^{-r_1}v_1, \dots, t^{-r_n}v_n$$

form a $k[t^{-1}]$ -basis of \mathcal{O}_V^∞ . We call the r_i the “scrollar invariants” of V and an accompanying basis v_1, \dots, v_n is called a “reduced basis”. The scrollar invariant 0 is realized if and only if $v_1 \in k$. Note that the scrollar invariants of K are just $\{0\} \cup \{e_1, \dots, e_{d-1}\}$, where e_i are the scrollar invariants of C with respect to φ .

We prove two auxiliary lemmas:

Lemma 23. *Let $V \subseteq L$ be a $k(t)$ -subspace with scrollar invariants r_1, \dots, r_n and reduced basis v_1, \dots, v_n . For every integer r define*

$$V^r = \{v \in \mathcal{O}_V \mid t^{-r}v \in \mathcal{O}_V^\infty\}$$

and let j_r be maximal such that $r_{j_r} \leq r$. Then V^r is a k -vector space with basis $\{t^i v_j\}_{j=1, \dots, j_r, i=0, \dots, r-j_r}$, and $k(t)V^r$ is a $k(t)$ -vector space with basis $\{v_1, \dots, v_{j_r}\}$.

Proof. It is clear that $\{t^i v_j\}_{j=1, \dots, n, i=0, 1, 2, \dots}$ is a k -basis for \mathcal{O}_V . When expanding $v \in \mathcal{O}_V$ with respect to this basis, the requirement $t^{-r}v \in \mathcal{O}_V^\infty$ is easily seen to be equivalent to the vanishing of the coordinates at $t^i v_j$ with $i + j_r - r > 0$, from which the first claim follows. The second claim is immediate from the first one. \square

The next lemma is frequently useful in proving that a candidate-reduced basis is indeed a reduced basis.

Lemma 24. *Let V be a $k(t)$ -vector subspace of L with scrollar invariants $r_1 \leq \dots \leq r_n$. Suppose that $v'_1, \dots, v'_n \in \mathcal{O}_V$ form a $k(t)$ -basis for V and that there are integers $r'_1 \leq \dots \leq r'_n$ such that*

$$r_1 + \dots + r_n = r'_1 + \dots + r'_n$$

and such that $t^{-r'_i} v'_i \in \mathcal{O}_V^\infty$. Then $r_i = r'_i$ for all i , and v'_1, \dots, v'_n is a reduced basis for V .

Proof. Let v_1, \dots, v_n be a reduced basis of V . Define the matrices

$$\begin{aligned} D_1 &= \text{diag}(t^{r_1}, \dots, t^{r_n}), \\ D_2 &= \text{diag}(t^{r'_1}, \dots, t^{r'_n}). \end{aligned}$$

Note that $\det D_1 = \det D_2 = t^{r_1 + \dots + r_n}$. Let B be the change of basis matrix from $\{v'_i\}_i$ to $\{v_i\}_i$. Since the v'_i are integral over $k[t]$, this matrix has entries in $k[t]$. We also have $\det B \neq 0$ since the v'_i form a $k(t)$ -basis of V . We make a similar reasoning above infinity. The change of basis matrix from $\{t^{-r'_i} v'_i\}_i$ to $\{t^{-r_i} v_i\}_i$ is given by $D_2 B D_1^{-1}$. It has entries in $k[t^{-1}]$ since $t^{-r'_i} v'_i$ is integral over $k[t^{-1}]$. It follows that $\det(D_2 B D_1^{-1}) = \det B \in k[t] \cap k[t^{-1}] = k$. Since $\det B$ is non-zero, we conclude that $\{v'_i\}_i$ is a reduced basis for V . \square

(4.5). **Scrollar invariants of representations and partitions.** We now introduce our scrollar invariants associated to irreducible representations. Recall that we can view L as the regular representation of S_d . We split L into isotypic components $W_\lambda \cong V_\lambda^{\dim V_\lambda}$ as in (1.3), one for each partition $\lambda \vdash d$. The patches \mathcal{O}_{W_λ} and $\mathcal{O}_{W_\lambda}^\infty$ then glue together to the vector bundle \mathcal{W}_λ from Section (1.3). The next corollaries to Lemma 23 establish the decomposition (3) and lie at the heart of everything that follows:

Corollary 25. *If $V \subseteq L$ is an irreducible subrepresentation then all scrollar invariants of V are equal to each other.*

Proof. Let v_1, \dots, v_n be a reduced basis for V and let $r_1 \leq \dots \leq r_n$ be the corresponding scrollar invariants. The k -vector space V^{r_1} from Lemma 23 has dimension j_{r_1} . But it contains all the conjugates of v_1 , and by the irreducibility of V we can find n conjugates which are linearly independent

over $k(t)$, so definitely over k . But then we must have $j_{r_1} = n$, i.e. $r_1 = r_2 = \dots = r_n$. \square

Corollary 26. *The scrollar invariants of $W_\lambda \subseteq L$ form a multi-set*

$e_{\lambda,1}, e_{\lambda,1}, \dots, e_{\lambda,1}, \quad e_{\lambda,2}, e_{\lambda,2}, \dots, e_{\lambda,2}, \quad \dots \quad e_{\lambda, \dim V_\lambda}, e_{\lambda, \dim V_\lambda}, \dots, e_{\lambda, \dim V_\lambda}$
with every block containing $\dim V_\lambda$ copies of the same entry.

Proof. Write $n = \dim V_\lambda$, let $v_{11}, \dots, v_{1n}, v_{21}, \dots, v_{2n}, \dots, v_{n1}, \dots, v_{nn}$ be a reduced basis of W_λ , and let

$$0 \leq r_{11} \leq \dots \leq r_{1n} \leq r_{21} \leq \dots \leq r_{2n} \leq \dots \leq r_{n1} \leq \dots \leq r_{nn}$$

be the corresponding scrollar invariants. Using the notation from Lemma 23, we know that $\dim k(t)W_\lambda^{-3} = 0$ and $\dim k(t)W_\lambda^{r_{nn}} = n^2$. By Lemma 23 it suffices to show that whenever

$$(19) \quad k(t)W_\lambda^{r-1} \subsetneq k(t)W_\lambda^r$$

for some $r = 0, \dots, r_{nn}$, the dimensions differ by a multiple of n . By assumption there exists a basis element v_{ij} with scrollar invariant exactly r . The conjugates of v_{ij} span a space $V_{ij} \cong V_\lambda$ of dimension n which is contained in $k(t)W_\lambda^r$ and which intersects $k(t)W_\lambda^{r-1}$ trivially. We can repeat this argument with $k(t)W_\lambda^{r-1}$ replaced by $k(t)W_\lambda^{r-1} + V_{ij}$ until (19) becomes an equality, at which moment the dimension has indeed increased with a multiple of n . \square

(4.6). **Resolvent curves and their scrollar invariants.** For every subgroup $H \subseteq S_d$ we call the fixed field L^H the “resolvent” of K with respect to H . Geometrically, this corresponds to a curve, denoted by $R_H C$, equipped with a morphism $R_H \varphi : R_H C \rightarrow \mathbf{P}^1$ of degree $[S_d : H]$. Its isomorphism class as a \mathbf{P}^1 -cover is only dependent on the conjugacy class of H . We now prove Theorem 4, expressing the scrollar invariants of L^H in terms of those of L .

Proof of Theorem 4. Fix a partition λ and write $n = \dim V_\lambda$ and $r = \text{mult}(V_\lambda, \text{Ind}_H^{S_d} \mathbf{1})$. By the proof of Corollary 26 we can write $W_\lambda = V_1 \oplus \dots \oplus V_n$ as an internal direct sum of irreducible subrepresentations $V_i \cong V_\lambda$, in such a way that a reduced basis of W_λ is obtained from reduced bases $\{v_{i1}, \dots, v_{in}\}$ of the V_i 's by simply taking the union. As before, we denote by $e_{\lambda,i}$ the unique scrollar invariant of V_i , which appears with multiplicity n . From Lemma 7 we see that the fixed subspaces V_i^H have dimension r , and by applying the lemma over k rather than $k(t)$, we find that V_i^H admits a $k(t)$ -basis consisting of k -linear combinations of the v_{ij} 's. It is immediate that this basis is reduced, with again $e_{\lambda,i}$ as the unique corresponding scrollar invariant, now appearing with multiplicity r . The union of these reduced bases is a reduced basis for $W_\lambda^H = V_1^H \oplus \dots \oplus V_n^H$ (e.g., because it extends to a reduced basis for W_λ), so the scrollar invariants of W_λ are obtained by considering r horizontal slices in (3).

We claim that taking the union over all partitions $\lambda \vdash d$ produces a reduced basis of L^H . For every partition λ of d , let $\{\omega_{\lambda,i}\}_i$ be the reduced basis of W_λ^H just constructed. (For some λ , this set may be empty.) This is clearly a $k(t)$ -basis for L^H . Let us first prove that it also concerns a $k[t]$ -basis of \mathcal{O}_{L^H} . Let $\omega \in \mathcal{O}_{L^H}$ and write

$$\omega = \sum_{\text{partitions } \lambda} \sum_i a_{\lambda,i} \omega_{\lambda,i},$$

for some $a_{\lambda,i} \in k(t)$; it suffices to prove that $a_{\lambda,i} \in k[t]$ for any λ, i . Fix λ and let $\rho \in Z(k[S_d])$ be the corresponding element from Lemma 8. As ρ is defined over k , it maps \mathcal{O}_L to itself and

$$\rho(\omega) = \sum_i a_{\lambda,i} \omega_{\lambda,i} \in \mathcal{O}_L \cap W_\lambda^H = \mathcal{O}_{W_\lambda}^H.$$

But the set $\{\omega_{\lambda,i}\}_i$ is a basis for \mathcal{O}_{W_λ} and so $a_{\lambda,i} \in k[t]$ for all i . An identical argument proves that the set $\{t^{-e_i^\lambda} \omega_{\lambda,i}\}_{\lambda,i}$ is a $k[t^{-1}]$ -basis for \mathcal{O}_L^∞ , with e_i^λ the scollar invariant associated with the basis element $\omega_{\lambda,i}$. \square

(4.7). Some first examples of partitions and their scollar invariants are: the trivial partition (d) , with scollar invariant $\{0\}$, and the partition $(d-1, 1)$, with scollar invariants $\{e_1, e_2, \dots, e_{d-1}\}$. To see the latter claim: one can view our given cover $\varphi : C \rightarrow \mathbf{P}^1$ as its own resolvent with respect to S_{d-1} , and then the claim follows from Theorem 4 along with

$$\text{Ind}_{S_{d-1}}^{S_d} \mathbf{1} \cong V_{(d)} \oplus V_{(d-1,1)}.$$

Alternatively, if $1, \alpha_1, \dots, \alpha_{d-1}$ is a reduced basis for \mathcal{O}_K then after a replacement of α_i with $\alpha_i - \frac{1}{d} \text{Tr}_{K/k(t)}(\alpha_i)$ we may assume that $\text{Tr}_{K/k(t)}(\alpha_i) = 0$. Then the $\alpha_1, \dots, \alpha_{d-1}$ form a reduced basis for $V_1 = W_{(d-1,1)} \cap K$ from which it follows that $(d-1, 1)$ has scollar invariants e_1, \dots, e_{d-1} in L .

Another basic case is the partition (1^d) , with scollar invariant $\{g' + 1\}$, where g' denotes the genus of $R_{A_d} C$. This again follows from Theorem 4, now using $\text{Ind}_{A_d}^{S_d} \mathbf{1} \cong V_{(d)} \oplus V_{(1^d)}$. From Theorem 27 below it will follow that $g' = g + d - 2$ as soon as φ is simply branched.

(4.8). **A genus formula for resolvent curves.** From this section onward we assume that the morphism $\varphi : C \rightarrow \mathbf{P}^1$ is simply branched, i.e. all non-trivial ramification is of the form $(2, 1^{d-2})$; recall that this implies that $\text{Gal}(K/k(t)) \cong S_d$. Under this assumption, we can express the genus of $R_H \varphi$ in terms of that of C . This may be known to specialists,⁸ but we could not find it explicitly in the existing literature, so let us include a proof.

Theorem 27. *Let $\varphi : C \rightarrow \mathbf{P}^1$ be a simply branched morphism of degree $d \geq 2$ over a field k with $\text{char } k = 0$ or $\text{char } k > 2$ and let H be a proper subgroup of S_d . The morphism $R_H \varphi : R_H C \rightarrow \mathbf{P}^1$ is branched over exactly the same*

⁸E.g., a special case is covered by [24, pp.111-113] and a number theoretic analogue is discussed in mathoverflow question 6674.

points as φ . Each such point ramifies with pattern $(2^{p(H)}, 1^{[S_d:H]-2p(H)})$, where

$$p(H) = (d-2)! \cdot \frac{|\{\text{transpositions } \sigma \in S_d \mid \sigma \notin H\}|}{|H|}.$$

In particular, the genus of $R_H C$ is $p(H)(g+d-1) + 1 - [S_d : H]$.

Note that the excluded case $H = S_d$ corresponds to the identity morphism $\mathbf{P}^1 \rightarrow \mathbf{P}^1$. Here $p(H) = 0$ so the theorem remains valid, except for the second sentence. At the other extreme, for $H = \{\text{id}\}$ which corresponds to the Galois closure $\overline{\varphi} : \overline{C} \rightarrow \mathbf{P}^1$, we have $p(H) = d!/2$. In this case the theorem says that all ramification patterns of $R_{\{\text{id}\}} \varphi$ are $(2^{d!/2})$.

Lemma 28. *Let H be a subgroup of S_d and let $\alpha \in L$ have stabilizer H under the Galois action. Let $\sigma \in S_d$ be a transposition. Then the number of ω in the orbit of α for which $\sigma(\omega) \neq \omega$ equals $2p(H)$.*

Proof. The map $S_d \rightarrow \{\text{transpositions in } S_d\} : \tau \mapsto \tau^{-1}\sigma\tau$ is surjective and $2(d-2)!$ -to-1. Therefore the number of $\tau \in S_d$ such that $\tau^{-1}\sigma\tau \notin H$ equals $2 \cdot |H| \cdot p(H)$. Thus the number of cosets $\tau H \in S_d/H$ for which $\sigma(\tau(\alpha)) \neq \tau(\alpha)$ equals $2p(H)$, from which the lemma follows. \square

Proof of Theorem 27. By Riemann–Hurwitz it suffices to prove that $R_H \varphi$ is branched over the same points as φ with the stated ramification patterns. Because these properties are local, we can work around $0 \in \mathbf{P}^1$ without loss of generality.

Let $\mathfrak{d}_K \subseteq k[t]$ denote the discriminant ideal of $K/k(t)$, i.e., the principal ideal generated by the field discriminant $\Delta_{K/k(t)}(\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_1, \alpha_2, \dots, \alpha_d$ denotes any $k[t]$ -basis of \mathcal{O}_K . Similarly consider the integral closure \mathcal{O}_{L^H} of $k[t]$ inside L^H along with its discriminant ideal $\mathfrak{d}_{L^H} \subseteq k[t]$. We will first show that

$$(20) \quad \mathfrak{d}_{L^H} = \mathfrak{d}_K^{p(H)}.$$

For this, we rely on a discriminant formula due to Lenstra, Pila and Pomerance [37, Thm. 4.4], which in our case states that

$$(21) \quad \mathfrak{d}_{L^H}^{d!} = \prod_{\sigma \in S_d \setminus \{\text{id}\}} \text{Norm}_{L/k(t)}(\mathfrak{J}_\sigma)^{|\{\tau : L^H \hookrightarrow L \mid \sigma \circ \tau \neq \tau\}|}$$

with \mathfrak{J}_σ denoting the ideal generated by all expressions of the form $\sigma(x) - x$ for $x \in \mathcal{O}_L$. Note that a non-zero prime ideal $\mathfrak{P} \subseteq \mathcal{O}_L$ divides \mathfrak{J}_σ if and only if σ is in the inertia group of \mathfrak{P} . Using that φ is simply branched, from [51, Sz. 1] we see that this inertia group is either trivial or consists of two elements. Therefore whenever σ is not a transposition, the corresponding factor in (21) contributes trivially. On the other hand, if σ is a transposition, then the number of embeddings $\tau : L^H \hookrightarrow L$ for which $\sigma \circ \tau \neq \tau$ equals $2p(H)$: this follows from Lemma 28 when applied to a primitive element ρ of L^H over $k(t)$, which can be chosen to be a polynomial expression in the

conjugates $\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(d)}$ of a primitive element $\alpha = \alpha^{(1)}$ of K over $k(t)$. We find that

$$\mathfrak{d}_{L^H}^{d!} = \prod_{\substack{\text{transpositions} \\ \sigma \in S_d}} \text{Norm}_{L/k(t)}(\mathfrak{I}_\sigma)^{2p(H)}.$$

This formula in combination with the same formula applied to the group $H = S_{d-1}$ of permutations fixing 1 yields (20).

It then follows from [39, III, Thm. 2.6 & 2.9] that $R_H \varphi$ is branched over the same points as φ . It also follows that, degree-wise, each branch point contributes $p(H)$ to the ramification divisor. Here we have used that $\text{char } k = 0$ or $\text{char } k > 2$, which ensures tame ramification. For $H = \{\text{id}\}$, corresponding to the Galois closure, we know that all ramification indices must be equal to each other. But since $p(H) = d!/2$ and the Galois closure has degree $d!$, we conclude that these ramification indices must in fact be 2. Consequently, the ramification indices must also be 1 or 2 for every resolvent cover, from which the theorem follows. \square

(4.9). **Volume and duality.** We prove two general facts about the scrollar invariants of a partition λ of d , still under the assumption that φ is simply branched. Firstly, we prove a closed formula for their sum

$$\text{vol}_K(\lambda) = e_{\lambda,1} + e_{\lambda,2} + \dots + e_{\lambda, \dim V_\lambda}.$$

We call this the “volume” of λ with respect to $K/k(t)$.⁹ This generalizes the well-known formula $g+d-1$ for the sum of the scrollar invariants e_1, \dots, e_{d-1} of C with respect to φ , i.e. the scrollar invariants of the partition $(d-1, 1)$.

Proposition 29 (volume formula). *Assume that $\varphi : C \rightarrow \mathbf{P}^1$ is simply branched. Let λ be a partition of d . Then*

$$\text{vol}_K(\lambda) = p(\lambda) \cdot (g + d - 1),$$

where $p(\lambda) = \frac{1}{2}(\dim V_\lambda - \chi_\lambda((12)))$.

The quantity $p(\lambda)$ admits several other characterizations:

$$p(\lambda) = \langle V_{(1^2)}, \text{Res}_{S_2}^{S_d} \lambda \rangle_{S_2} = \text{mult}(\lambda, \text{Ind}_{S_2}^{S_d} V_{(1^2)}).$$

One can also verify the inductive formula $p(\lambda) = \sum_{\lambda' < \lambda} p(\lambda')$ for $d \geq 3$, where the sum is over all partitions of $d-1$ which are smaller than λ (i.e., which are obtained from λ by removing a box from the Young diagram).

Proof of Proposition 29. We claim that

$$\sum_{\lambda \in \text{Ind}_H^{S_d} \mathbf{1}} \text{vol}_K(\lambda) = p(H)(g + d - 1).$$

⁹This is in view of Minkowski’s second theorem and the analogy between the scrollar invariants and the successive minima of the Minkowski lattice attached to a number field [30, §7]; in fact calling it the “log-volume” would make the analogy more precise.

for each subgroup $H \subseteq S_d$, where the sum is understood to run over all partitions λ of d , counted with multiplicity $\text{mult}(V_\lambda, \text{Ind}_H^{S_d} \mathbf{1})$. Assuming the claim, we note that

$$p(H) = \frac{1}{2}(\dim \text{Ind}_H^{S_d} \mathbf{1} - \chi_{\text{Ind}_H^{S_d} \mathbf{1}}((12))),$$

therefore

$$p(H) = \sum_{\lambda \in \text{Ind}_H^{S_d} \mathbf{1}} p(\lambda),$$

allowing us to conclude that the linear system of equations

$$\sum_{\lambda \in \text{Ind}_H^{S_d} \mathbf{1}} X_\lambda = \sum_{\lambda \in \text{Ind}_H^{S_d} \mathbf{1}} p(\lambda)(g + d - 1)$$

(one equation for each $H \subseteq S_d$) in the variables X_λ (one variable for each $\lambda \vdash d$) admits the solution $X_\lambda = \text{vol}_K(\lambda)$. But, clearly, also $X_\lambda = p(\lambda)(g + d - 1)$ is a solution, so it suffices to see that the solution is unique. For this, restrict to those equations for which $H = S_\rho$ is a Young subgroup, with ρ a partition of d . By [44, p. 88] the resulting coefficient matrix is upper triangular (with respect to a certain ordering on all partitions) and all diagonal elements are 1. Hence our system indeed has a unique solution, as wanted.

To prove the claim, we consider a reduced basis $\{\omega_i^\lambda\}_{\lambda, i}$ of $L^H/k(t)$ as in the proof of Theorem 4. That is: for every fixed partition λ of d the elements ω_i^λ , $i = 1, \dots, r \dim V_\lambda$ form a reduced basis of W_λ , and the corresponding scollar invariants e_i^λ are the $e_{\lambda, j}$'s, each counted with multiplicity r . The discriminant ideal $\mathfrak{d}_{L^H} \subseteq k[t]$ is generated by $\det T$, where $T = (\text{Tr}_{L/k(t)}(\omega_i^\lambda \omega_j^{\lambda'}))_{\lambda, \lambda', i, j}$. If $\lambda \neq \lambda'$ are two partitions of S_d then $\text{Tr}_{L/k(t)}(\omega_i^\lambda \omega_j^{\lambda'}) = 0$, as $V_\lambda \otimes V_{\lambda'}$ does not contain a copy of the trivial representation. Thus T falls apart into the block matrices $T_\lambda = (\text{Tr}_{L/k(t)}(\omega_i^\lambda \omega_j^\lambda))_{i, j}$. We obtain that

$$\mathfrak{d}_K^{p(H)} = \mathfrak{d}_{L^H} = (\det T) = \left(\prod_{\lambda} \det T_\lambda \right)$$

where the first equality follows from the proof of Theorem 27; note that this uses the simple branching assumption (also note that this equality remains true for $H = S_d$). We now repeat this argument above infinity, where we write $\mathfrak{d}_K^\infty, \mathfrak{d}_{L^H}^\infty \subseteq k[t^{-1}]$ to denote the discriminant ideals of $\mathcal{O}_K^\infty, \mathcal{O}_{L^H}^\infty$. Since the ω_i^λ 's form a reduced basis, the corresponding matrices T_λ^∞ are equal to

$$(\text{Tr}_{L/k(t)}(t^{-e_i^\lambda - e_j^\lambda} \omega_i^\lambda \omega_j^\lambda))_{i, j}.$$

We similarly find

$$(\mathfrak{d}_K^\infty)^{p(H)} = \mathfrak{d}_{L^H}^\infty = \left(\prod_{\lambda} \det T_\lambda^\infty \right) = \left(\prod_{\lambda} t^{-2 \sum_i e_i^\lambda} \det T_\lambda \right).$$

Using that a generator of \mathfrak{d}_K^∞ is obtained from a generator of \mathfrak{d}_K by scaling with $t^{-2(g+d-1)}$ we find that

$$-2p(H)(g+d-1) = \sum_{\lambda} -2 \sum_i e_i^\lambda = -2 \sum_{\lambda \in \text{Ind}_H^{S_d} \mathbf{1}} e_{\lambda,j}$$

from which the claim follows. \square

(4.10). Secondly, we can explicitly relate the scollar invariants of a partition λ to those of the dual partition λ^* , which is obtained by transposing the corresponding Young diagram.

Proposition 30 (duality). *Let λ be a partition of d and consider its multi-set of scollar invariants $\{e_{\lambda,i}\}_i$. Then the multi-set of scollar invariants of λ^* is given by $\{g+d-1-e_{\lambda,i}\}_i$.*

Proof. It suffices to show that the scollar invariants of W_λ and W_{λ^*} are obtained from each other through the map $e \mapsto g+d-1-e$. Let ω_i , $i = 1, \dots, (\dim V_\lambda)^2$, be a reduced basis for W_λ with corresponding scollar invariants $e_{\lambda,i}$ and let δ be a generator of $W_{(1^d)}$ as in (3.2). By Theorem 27 the scollar invariant of $W_{(1^d)}$ is $g+d-1$. Let $\mathcal{D}_{L/k(t)}$ denote the different ideal of the extension $L/k(t)$ and note that this is just the principal ideal generated by δ because all branch points have ramification pattern $(2^{d/2})$ by Theorem 27. Let ω_i^* be the dual basis of W_λ with respect to the trace pairing, i.e., $\text{Tr}_{L/k(t)}(\omega_i \omega_j^*) = \delta_{ij}$ where δ_{ij} is the Kronecker delta. We claim that the elements $\delta \omega_i^*$ form a reduced basis of W_{λ^*} . To prove this, we use Lemma 24. First of all, note that the elements

$$\delta \omega_i^* \in \delta \mathcal{D}_{L/k(t)}^{-1} = \mathcal{O}_L$$

are integral over $k[t]$. Also, it is clear that they form a $k(t)$ -basis for W_{λ^*} . Playing the same game above infinity, so starting with $t^{-e_{\lambda,i}} \omega_i$, we find the elements $t^{-(g+d-1-e_{\lambda,i})-2} \delta \omega_i^*$, which are all integral over $k[t^{-1}]$. Using Proposition 29 for λ , we compute

$$\sum_i (g+d-1-e_{\lambda,i}) = \frac{1}{2}(\dim V_\lambda + \chi_\lambda((12)))(g+d-1),$$

from which the desired result follows, as $\dim V_\lambda = \dim V_{\lambda^*}$ and $\chi_{\lambda^*}((12)) = -\chi_\lambda((12))$. \square

(4.11). **Scollar invariants of hooks.** Recall from (4.7) that the scollar invariants of the standard partition $(d-1, 1)$ are simply the scollar invariants e_1, \dots, e_{d-1} of our input map $\varphi : C \rightarrow \mathbf{P}^1$, and that the sign representation (1^d) has $g+d-1$ as its unique scollar invariant. These facts are part of the more general statement from Proposition 2, which we now prove:

Proof of Proposition 2. Note that indeed

$$\dim V_\lambda = \binom{d-1}{i}$$

is the number of i -element subsets of $\{1, 2, \dots, d-1\}$. Take a reduced basis $\alpha_1, \dots, \alpha_{d-1}$ of $V_1 = W_{(d-1,1)} \cap L^{S_{d-1}}$, with scrollar invariants e_1, \dots, e_{d-1} . Then $\{\alpha_j^{(m)} \mid 1 \leq j, m \leq d-1\}$ is a reduced basis for $W_{(d-1,1)}$. The determinant of the matrix

$$D = \begin{pmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} & \dots & \alpha_1^{(d-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{d-1}^{(1)} & \alpha_{d-1}^{(2)} & \dots & \alpha_{d-1}^{(d-1)} \end{pmatrix}$$

appears, up to sign, as a summand in the expansion of the determinant of (15) along the first row. Since the latter determinant is non-zero and all these summands are conjugate to each other, we see that $\det D \neq 0$. We claim that the $i \times i$ minors of D form a reduced basis for W_λ . First note that acting with S_d on such a minor generates a representation isomorphic to V_λ , so it is indeed contained in W_λ . Secondly, we have

$$\dim W_\lambda = (\dim V_\lambda)^2 = \binom{d-1}{i}^2$$

minors and they are linearly independent because the $(d-1) \times (d-1)$ matrix whose entries are these minors has determinant

$$\det(D)^{\binom{d-2}{i-1}} \neq 0$$

by the Sylvester–Franke theorem. Thus our minors form a basis of W_λ , and clearly they are all integral over $k[t]$. Now repeat this construction starting from the basis $t^{-e_1}\alpha_1, \dots, t^{-e_{d-1}}\alpha_{d-1}$, in order to obtain another basis of W_λ consisting of $i \times i$ minors which are integral over $k[t^{-1}]$. The statement then follows through an application of Lemma 24. \square

(4.12). As a first non-trivial application of the work done above, we can describe the scrollar invariants of $\mathbf{R}_{A_{d-1}}\varphi : \mathbf{R}_{A_{d-1}}C \rightarrow \mathbf{P}^1$, where A_{d-1} denotes the subgroup of S_d consisting of all even permutations fixing 1.

Corollary 31. *The scrollar invariants of the curve $\mathbf{R}_{A_{d-1}}C$ with respect to $\mathbf{R}_{A_{d-1}}\varphi$ are $e_1, \dots, e_{d-1}, g + d - 1 - e_1, \dots, g + d - 1 - e_{d-1}$ and $g + d - 1$.*

Proof. We have that $\text{Ind}_{A_{d-1}}^{S_d} \mathbf{1} = V_{(d)} \oplus V_{(d-1,1)} \oplus V_{(2,1^{d-2})} \oplus V_{(1^d)}$. Thus the result follows from Proposition 30 (or from Proposition 2), together with Theorem 4. \square

We believe that, as soon as $d \geq 4$, the only resolvent covers whose scrollar invariants admit a description completely in terms of the scrollar invariants of $\varphi : C \rightarrow \mathbf{P}^1$ are those with respect to S_{d-1}, A_{d-1}, A_d (and the degenerate case S_d), although we do not have a proof for this.

(4.13). **The case of $(3, 1^{d-3})$ ramification.** For use in Section 5 we extend our proof of the volume formula for the partition $(d-2, 2)$ such that it also covers the case of $(3, 1^{d-3})$ ramification. This allows for a slight generalization of our results. We say that a degree $d \geq 4$ morphism $\varphi : C \rightarrow \mathbf{P}^1$ has “good ramification” if all non-trivial ramification is of type $(2, 1^{d-2})$ or $(3, 1^{d-3})$ and the Galois closure of φ has the full symmetric group S_d as its Galois group; the latter property no longer follows automatically.

Lemma 32. *If φ has good ramification, then*

$$\text{vol}_K((d-2, 2)) = p((d-2, 2))(g + d - 1).$$

Proof. We start by modifying the proof of Theorem 27 for the subgroup $H = S_2 \times S_{d-2}$, in such a way that it covers the case of good ramification, rather than just simple branching. Remember that this proof boiled down to showing that

$$|\{\tau : L^H \hookrightarrow L \mid \sigma \circ \tau \neq \tau\}| = p(H) \cdot |\{\tau : L^{S_{d-1}} \hookrightarrow L \mid \sigma \circ \tau \neq \tau\}|$$

for every transposition $\sigma \in S_d$. It suffices to show that this is also true in the case where σ is a 3-cycle, in which case the right-hand side becomes $3p(H) = 3(d-2)$. Let us rewrite the left-hand side as

$$\frac{|\{\tau \in S_d \mid \tau^{-1}\sigma\tau \notin H\}|}{|H|} = \chi_{\text{Ind}_H^{S_d} \mathbf{1}}(\text{id}) - \chi_{\text{Ind}_H^{S_d} \mathbf{1}}(\sigma).$$

Since we know that $\text{Ind}_H^{S_d} \mathbf{1} \cong V_{(d)} \oplus V_{(d-1,1)} \oplus V_{(d-2,2)}$ it suffices to prove that $\beta_1 - \chi_{(d-2,2)}(\sigma) = 3(d-3)$ for every 3-cycle σ . The character $\chi_{(d-2,2)}(\sigma)$ can be computed using the Murnaghan–Nakayama rule [44, Thm. 4.10.2] and is found to be equal to

$$\chi_{(d-2,2)}(\sigma) = \begin{cases} -1 & \text{if } d = 4, 5, \\ 0 & \text{if } d = 6, \\ \dim V_{(d-5,2)} & \text{if } d \geq 7. \end{cases}$$

A small calculation then proves the desired genus formula for L^H .

To conclude, the proof of Proposition 29 shows that

$$\sum_{\lambda \in \text{Ind}_H^{S_d} \mathbf{1}} \text{vol}_K(\lambda) = p(H)(g + d - 1),$$

so the result follows from the fact that $\text{vol}_K((d-1, 1)) = g + d - 1$ and $\text{Ind}_H^{S_d} \mathbf{1} \cong V_{(d)} \oplus V_{(d-1,1)} \oplus V_{(d-2,2)}$. \square

In general, we expect that for every partition $\lambda \vdash d$ there is a list of “allowed” ramification patterns, such that Proposition 29 holds. For $\lambda = (d-1, 1)$ any ramification type is allowed, while for $\lambda = (d-2, 2)$ we have just proven that $(2, 1^{d-2})$ and $(3, 1^{d-3})$ are both good. We expect that no other non-trivial ramification is good for $(d-2, 2)$. We will say more about this in (7.7)–(7.8).

5. SCHREYER'S INVARIANTS ARE SCROLLAR

(5.1). This section is devoted to proving Theorem 3, reinterpreting the Schreyer invariants of a simply branched degree $d \geq 4$ morphism $\varphi : C \rightarrow \mathbf{P}^1$ as scollar invariants of the partitions $\lambda_{i+1} = (d - i - 1, 2, 1^{i-1})$, where $i = 1, \dots, d - 3$. The proof can be found in (5.5). We begin with gathering some facts on the relative canonical embedding.

(5.2). **Defining equations of C inside $\mathbf{P}(\mathcal{E})$ and their syzygies.** We know from (7) that the relative canonical embedding realizes C inside $\mathbf{P}(\mathcal{E})$ as the intersection of β_1 divisors

$$(22) \quad D_j \in 2H - b_j^{(1)}R$$

and this is a minimal set of generators. Equipping the fibers of our \mathbf{P}^{d-2} -bundle $\pi : \mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}^1$ with homogeneous coordinates x_1, x_2, \dots, x_{d-1} and similarly providing \mathbf{P}^1 with homogeneous coordinates s, t , it makes sense to talk about defining equations. Being in the class (22) then amounts to being defined by a quadratic form

$$(23) \quad \sum_{j_1+j_2+\dots+j_{d-1}=2} \varphi_{j_1, j_2, \dots, j_{d-1}} \cdot x_1^{j_1} x_2^{j_2} \cdots x_{d-1}^{j_{d-1}}$$

where each coefficient $\varphi_{j_1, j_2, \dots, j_{d-1}} \in k[s, t]$ is homogeneous of degree

$$(24) \quad j_1 e_1 + j_2 e_2 + \dots + j_{d-1} e_{d-1} - b_j^{(1)}$$

(the coefficient is zero if this quantity is negative). The morphism φ is just the restriction of π to C , i.e., it amounts to projection on the (s, t) -coordinates.

The next steps of the resolution (7) can be made explicit as well, in terms of syzygies. Concretely, for $1 \leq i \leq d - 4$ the map

$$\bigoplus_{j=1}^{\beta_{i+1}} \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-(i+2)H + b_j^{(i+1)}R) \rightarrow \bigoplus_{j=1}^{\beta_i} \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-(i+1)H + b_j^{(i)}R)$$

can be represented by a $\beta_i \times \beta_{i+1}$ matrix whose entry on row j_1 and column j_2 is a linear form $\varphi_1 x_1 + \varphi_2 x_2 + \dots + \varphi_{d-1} x_{d-1}$ where each $\varphi_\ell \in k[s, t]$ is homogeneous of degree

$$e_\ell + b_{j_1}^{(i)} - b_{j_2}^{(i+1)}.$$

The last step of the resolution is then again described by $\beta_{d-3} = \beta_1$ quadratic forms as in (23), where now the coefficients $\varphi_{j_1, j_2, \dots, j_{d-1}} \in k[s, t]$ are seen to be homogeneous of degree

$$j_1 e_1 + j_2 e_2 + \dots + j_{d-1} e_{d-1} + b_j^{(d-3)} + d + 1 - g,$$

which in fact just equals (24) for an appropriate ordering of the Schreyer invariants, because of duality: see (26) below.

(5.3). The geometric generic fiber of φ is the configuration of d points in \mathbf{P}^{d-2} over $k(t)$ cut out by the quadrics

$$\sum_{j_1+j_2+\dots+j_{d-1}=2} \varphi_{j_1, j_2, \dots, j_{d-1}}(1, t) \cdot x_1^{j_1} x_2^{j_2} \cdots x_{d-1}^{j_{d-1}} \in k(t)[x_1, x_2, \dots, x_{d-1}]$$

for $j = 1, 2, \dots, \beta_1$. Let us become more precise about our claim from (1.9) that this geometric generic fiber is in fact a point configuration of Bhargava type.

Proposition 33. *Consider a morphism $\varphi : C \rightarrow \mathbf{P}^1$ of degree $d \geq 3$ and consider a reduced basis $1, \alpha_1, \dots, \alpha_{d-1}$ of the corresponding function field extension $k(t) \subseteq k(C)$. The geometric generic fiber of φ , when viewed as a configuration of d points in \mathbf{P}^{d-2} through the relative canonical embedding, can be identified with Bhargava's point set*

$$[\alpha_1^{*(1)} : \dots : \alpha_{d-1}^{*(1)}], [\alpha_1^{*(2)} : \dots : \alpha_{d-1}^{*(2)}], \dots, [\alpha_1^{*(d)} : \dots : \alpha_{d-1}^{*(d)}]$$

attached to $1, \alpha_1, \dots, \alpha_{d-1}$ as in (1.8).

Proof. Recall from (4.2) that our reduced basis $1, \alpha_1, \dots, \alpha_{d-1}$ is a $k[t]$ -basis of $\mathcal{O}_{k(C)}$ such that $1, t^{-e_1}\alpha_1, \dots, t^{-e_{d-1}}\alpha_{d-1}$ is a basis of $\mathcal{O}_{k(C)}^\infty$ over $k[t^{-1}]$, and that this property is tantamount to the decomposition

$$\varphi_*\mathcal{O}_C \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{E}^\vee, \quad \mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_{d-1})$$

from (1). Also, remember from (4.7) that the α_i can be assumed to have trace zero, i.e., they form a reduced basis of V_1 . Then the dual basis is of the form $1, \alpha_1^*, \dots, \alpha_{d-1}^*$. We claim that this dual basis plays a similar role in the decomposition

$$\varphi_*\mathcal{O}_C(K_{C/\mathbf{P}^1}) \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{E}$$

with K_{C/\mathbf{P}^1} the ramification divisor of φ (this is a relative canonical divisor). Indeed, the explicit formulae $\alpha_i^* = \det D_{i+1,1}/\det D$ from (3.9) imply that $1, \alpha_1^*, \dots, \alpha_{d-1}^*$ forms a $k[t]$ -basis of $\mathcal{O}_C(K_{C/\mathbf{P}^1})(\mathbf{A}^1)$, while the counterparts of these formulae above infinity show that $1, t^{e_1}\alpha_1^*, \dots, t^{e_{d-1}}\alpha_{d-1}^*$ is a $k[t^{-1}]$ -basis of $\mathcal{O}_C(K_{C/\mathbf{P}^1})(\mathbf{P}^1 \setminus \{0\})$.¹⁰

The relative canonical embedding is the map $\iota : C \hookrightarrow \mathbf{P}(\mathcal{E})$ induced by the morphism $\varphi^*\mathcal{E} \rightarrow \varphi^*\varphi_*\mathcal{O}_C(K_{C/\mathbf{P}^1}) \rightarrow \mathcal{O}_C(K_{C/\mathbf{P}^1})$ (which turns out to be surjective) coming from the natural inclusion $\mathcal{E} \rightarrow \varphi_*\mathcal{O}_C(K_{C/\mathbf{P}^1})$; see Casnati–Ekedahl [12, Thm. 2.1] or the recent treatment due to Landesman–Vakil–Wood [36, §3.1] for details. In view of the above, in explicit terms this means that on the patch above \mathbf{A}^1 , the morphism ι is given by

$$\varphi^{-1}(\mathbf{A}^1) \rightarrow \mathbf{A}^1 \times \mathbf{P}^{d-2} : p \mapsto (\varphi(p), [\alpha_1^*(p) : \alpha_2^*(p) : \dots : \alpha_{d-1}^*(p)]),$$

¹⁰The same reasoning for the absolute canonical divisor $K_C = K_{C/\mathbf{P}^1} - 2(t)_\infty = (dt)$ shows that $1, \alpha_1^*, \dots, \alpha_{d-1}^*$ and $t^{-2}, t^{e_1-2}\alpha_1^*, \dots, t^{e_{d-1}-2}\alpha_{d-1}^*$ are respective bases for $\mathcal{O}_C(K_C)(\mathbf{A}^1)$ and $\mathcal{O}_C(K_C)(\mathbf{P}^1 \setminus \{0\})$. This yields the decomposition $\varphi_*\mathcal{O}_C(K_C) \cong \mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{E}(-2)$ mentioned in (1.2).

from which the proposition follows. \square

(5.4). **Schreyer's invariants are scollar.** A key role in the proof of Theorem 3 will be played by the following formula, saying that the volume of the partition λ_{i+1} is as it should be:

Lemma 34. *Let $1 \leq i \leq d - 3$. Then we have that*

$$\text{vol}_{k(C)}(\lambda_{i+1}) = \sum_{j=1}^{\beta_i} b_j^{(i)}.$$

Proof. Firstly, we claim that $p(\lambda_{i+1}) = (d - 2 - i) \binom{d-2}{i-1}$. To prove this claim, define

$$\beta_i^{(e)} = \frac{e}{i+1} (e - 2 - i) \binom{e-2}{i-1}.$$

Using the Murnaghan–Nakayama rule [44, Thm. 4.10.2], one computes that

$$\chi_{(d-2,2)}((1\ 2)) = \begin{cases} 0 & d = 4, \\ 1 & d = 5, \\ 1 + \beta_1^{(d-2)} & d \geq 6, \end{cases} \quad \chi_{(d-3,2,1)}((1\ 2)) = \begin{cases} -1 & d = 5, \\ 0 & d = 6, \\ \beta_2^{(d-2)} & d \geq 7, \end{cases}$$

while for $3 \leq i \leq d - 5$ we have $\chi_{\lambda_{i+1}}((1\ 2)) = \beta_i^{(d-2)} - \beta_{i-2}^{(d-2)}$. The values of $\chi_{\lambda_{d-3}}((1\ 2))$ and $\chi_{\lambda_{d-2}}((1\ 2))$ can be determined by duality. The claim follows by an explicit calculation.

On the other hand, by [8, Prop. 2.9] we have that

$$(25) \quad \sum_{j=1}^{\beta_i} b_j^{(i)} = (d - 2 - i) \binom{d-2}{i-1} (g + d - 1),$$

where we caution the reader that [8, Prop. 2.9] uses the different convention for the $b_j^{(i)}$ that was discussed in (1.11). So the lemma follows from Proposition 29. \square

(5.5). We are now ready to prove Theorem 3.

Proof of Theorem 3. For $i = 1, \dots, d - 3$, write $e_1^{(i)} \leq e_2^{(i)} \leq \dots \leq e_{\beta_i}^{(i)}$ for the scollar invariants of λ_{i+1} with respect to φ , and likewise sort the Schreyer invariants such that

$$b_1^{(i)} \leq b_2^{(i)} \leq \dots \leq b_{\beta_i}^{(i)}.$$

Our aim is to prove that $e_j^{(i)} = b_j^{(i)}$ for all i and j .

We follow the notation from Section 3. In regards to bases, we now assume that $\alpha_1, \dots, \alpha_{d-1}$ is a reduced basis for V_1 and similarly that $\omega_1^i, \dots, \omega_{\beta_{i-1}}^i$ is a reduced basis for V_i , $i = 2, \dots, d - 2$. Note that the scollar invariants of V_1 are simply the scollar invariants e_1, \dots, e_{d-1} of C with respect to φ , while those of V_i are

$$e_1^{(i-1)}, e_2^{(i-1)}, \dots, e_{\beta_{i-1}}^{(i-1)}$$

for $i = 2, \dots, d-2$. This follows, for instance, from the proof of Theorem 4.

Applying the construction from Section 3, in view of Proposition 33 we obtain a minimal graded free resolution

$$\dots \rightarrow V_4^* \otimes R(-4) \xrightarrow{\psi_3^*} V_3^* \otimes R(-3) \xrightarrow{\psi_2^*} V_2^* \otimes R(-2) \xrightarrow{\psi_1^*} R \rightarrow R/I \rightarrow 0.$$

of the homogeneous coordinate ring of the geometric generic fiber of φ as a graded module over the polynomial ring $R = k(t)[x_1, \dots, x_{d-1}]$. We can assume that the polynomials p^1, \dots, p^{d-3} , used to define the maps

$$\psi_1 : \text{Sym}^2 V_1 \rightarrow V_2, \quad \psi_i : V_i \otimes V_1 \rightarrow V_{i+1}, i = 2, 3, \dots, d-3,$$

have coefficients in k : indeed, we can find them using representation theory over k , rather than $k(t)$. Consequently, the maps $\psi_1, \dots, \psi_{d-3}$ send tensors of elements that are integral over $k[t]$, resp. $k[t^{-1}]$, to elements that are integral over $k[t]$, resp. $k[t^{-1}]$.

Let us first discuss the implications for ψ_1 . Because $\alpha_1, \dots, \alpha_{d-1}$ and $\omega_1^2, \dots, \omega_{\beta_1}^2$ are $k[t]$ -bases we must have

$$\psi_1(\alpha_m \otimes \alpha_n) = \sum_{\ell=1}^{\beta_1} Q_{mn}^\ell \omega_\ell^2,$$

with $Q_{mn}^\ell \in k[t]$, for all $1 \leq m \leq n \leq d-1$. By linearity, it follows that

$$\psi_1(t^{-e_m} \alpha_m \otimes t^{-e_n} \alpha_n) = \sum_{\ell=1}^{\beta_1} t^{-e_m - e_n + e_\ell^{(1)}} Q_{mn}^\ell (t^{-e_\ell^{(1)}} \omega_\ell^2).$$

But $t^{-e_1} \alpha_1, \dots, t^{-e_{d-1}} \alpha_{d-1}$ and $t^{-e_1^{(1)}} \omega_1^2, \dots, t^{-e_{\beta_1}^{(1)}} \omega_{\beta_1}^2$ are $k[t^{-1}]$ -bases, so by our reasoning we must also have

$$t^{-e_m - e_n + e_\ell^{(1)}} Q_{mn}^\ell \in k[t^{-1}]$$

for all ℓ , which shows that

$$\deg Q_{mn}^\ell \leq e_m + e_n - e_\ell^{(1)},$$

or in other words that the quadric Q^ℓ defines an element of $2H - e_\ell^{(1)}R$ containing C . By the minimality of (7) we must have

$$e_\ell^{(1)} \leq b_\ell^{(1)}$$

for all ℓ . But from Lemma 34 we already know that

$$\sum_{\ell=1}^{\beta_1} e_\ell^{(1)} = \sum_{\ell=1}^{\beta_1} b_\ell^{(1)},$$

from which Theorem 3 follows in the case $i = 1$.

The idea for the following steps is similar. Let $2 \leq i \leq d - 3$. By our choice of bases we can write

$$\psi_i(\omega_m^i \otimes \alpha_n) = \sum_{\ell=1}^{\beta_i} L_{mn}^\ell \omega_m^{i+1},$$

for certain $L_{mn}^\ell \in k[t]$. Above infinity we find

$$\psi_i(t^{-e_m^{(i-1)}} \omega_m^i \otimes t^{-e_n} \alpha_n) = \sum_{\ell=1}^{\beta_i} t^{-e_m^{(i-1)} - e_n + e_\ell^{(i)}} L_{mn}^\ell (t^{-e_\ell^{(i)}} \omega_\ell^{i+1})$$

and we conclude as above that

$$t^{-e_m^{(i-1)} - e_n + e_\ell^{(i)}} L_{mn}^\ell \in k[t^{-1}],$$

or in other words that

$$\deg L_{mn}^\ell \leq e_m^{(i-1)} + e_n - e_\ell^{(i)} = b_m^{(i-1)} + e_n - e_\ell^{(i)},$$

where the equality follows by induction on i . Again, the minimality of (7) implies that

$$e_\ell^{(i)} \leq b_\ell^{(i)},$$

which in view of Lemma 34 allows us to conclude that $b_\ell^{(i)} = e_\ell^{(i)}$. \square

(5.6). A well-known consequence to the self-duality of the relative minimal resolution, see [12, Thm. 1.3] or [45, Cor. 4.4], reads that

$$(26) \quad \left\{ b_j^{(d-2-i)} \right\}_j = \left\{ g + d - 1 - b_j^{(i)} \right\}_j$$

as multi-sets. Assuming simple branching, one can view this as a special case of Proposition 30. Indeed, because the partitions λ_{i+1} and λ_{d-i-1} are dual to each other, i.e., they are obtained from one another by transposing Young diagrams, this proposition along with Theorem 3 immediately implies the duality statement.

(5.7). We end this section with a proof of our exemplary Theorem 5:

Proof of Theorem 5. This follows from the decomposition

$$\mathrm{Ind}_{S_2 \times S_d}^{S_d} \mathbf{1} \cong V_{(d)} \oplus V_{(d-1,1)} \oplus V_{(d-2,2)}.$$

along with Proposition 2, Theorem 3 and Theorem 4. \square

(5.8). The case $i = 1$ of Theorem 3 remains true under the weaker assumption of good ramification, as defined in (4.13).

Proposition 35 (addendum to Theorem 3). *Consider a degree $d \geq 4$ cover $C \rightarrow \mathbf{P}^1$ with good ramification. The multi-set of scrollar invariants of $(d - 2, 2)$ with respect to φ is given by*

$$\{ b_j \}_{j=1,2,\dots,\beta_1},$$

the splitting type of the first syzygy bundle in the relative minimal resolution of C with respect to φ .

Proof. This is an exact copy of the proof of Theorem 3, except that now one needs to use Lemma 32 as a substitute for Proposition 29 in establishing Lemma 34. \square

6. EXAMPLES

(6.1). **Low degree examples.** We have made a repository containing explicit descriptions of the scrollar invariants of any resolvent of any simply branched morphism $\varphi : C \rightarrow \mathbf{P}^1$ of degree $d \leq 6$. It can be found at <https://homes.esat.kuleuven.be/~wcastryc/>. Let us highlight three examples: Lagrange’s cubic resolvent in degree $d = 4$, Cayley’s sextic resolvent in degree $d = 5$, and an “exotic resolvent” in degree $d = 6$.

(6.2). Up to conjugation, $D_4 = \langle (12), (1324) \rangle$ is the unique subgroup of S_4 which is isomorphic to the dihedral group of order 8. In Galois theory, the resolvent of a quartic polynomial $f(x)$ with respect to D_4 is also known as “Lagrange’s cubic resolvent” and is best known as a tool for solving $f(x) = 0$ in terms of radicals. When applied to a simply branched degree 4 cover of \mathbf{P}^1 by a curve of genus g , the resolvent construction results in a degree 3 cover of \mathbf{P}^1 by a curve of genus $g + 1$, having the following scrollar invariants:

Corollary 36 (Casnati). *Consider a degree 4 simply branched cover $\varphi : C \rightarrow \mathbf{P}^1$ having Schreyer invariants b_1, b_2 . The scrollar invariants of $\mathbf{R}_{D_4} \varphi : \mathbf{R}_{D_4} C \rightarrow \mathbf{P}^1$ are equal to the Schreyer invariants of C with respect to φ .*

Proof. One checks that

$$\mathrm{Ind}_{D_4}^{S_4} \mathbf{1} \cong V_{(4)} \oplus V_{(2^2)}$$

so, in the case where φ is simply branched, this is an immediate consequence of Theorem 3 and Theorem 4. \square

Both Casnati [11, Def. 6.3-6.4] and Deopurkar–Patel [21, Prop. 4.6] use the language of Recillas’ trigonal construction [42], rather than the cubic resolvent. This construction applies to arbitrary degree 4 covers and produces smooth and geometrically integral curves as soon as there is no ramification of type (2^2) or (4) , even in the case of a smaller Galois group (the only other option being A_4). Therefore Casnati’s result is slightly stronger than Corollary 36. Note that the case of an S_4 -cover without ramification of type (2^2) or (4) can in fact be settled by invoking Proposition 35 instead of Theorem 3.

(6.3). For the sake of exposition, let us briefly recall the connection between Lagrange’s cubic resolvent and Recillas’ trigonal construction. This can be found in e.g. [52, §8.6], [50] or [42]. Recall that our curve C arises as the complete intersection of two divisors

$$D_1 \in 2H - b_1R \quad \text{and} \quad D_2 \in 2H - b_2R$$

inside $\mathbf{P}(\mathcal{E})$. Every fiber of the bundle map $\pi : \mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}^1$ is a \mathbf{P}^2 , in which our curve cuts out 4 points, counting multiplicities. Given a configuration of

4 generally positioned points in \mathbf{P}^2 , there are three ways in which these can be viewed as a union of two pairs. If each time we take the intersection point of the two lines spanned by these pairs then we find a “dual” configuration of 3 points in the same \mathbf{P}^2 , see Figure 2. By applying this procedure, or rather



FIGURE 2. “Dual” points associated with 4 points in \mathbf{P}^2 in general position.

a scheme-theoretic version of it (to cope with multiplicities), to the fibers of φ in $\mathbf{P}(\mathcal{E})$ we find a family of point triples that swipe out a new curve C' which naturally comes equipped with a degree 3 map $\varphi' : C' \rightarrow \mathbf{P}^1$; whence the trigonal construction. If we let $A_1, A_2 \in k[s, t]^{3 \times 3}$ denote symmetric matrices with homogeneous entries corresponding to the ternary quadratic forms defining D_1 and D_2 , then it is easy to see that

$$(27) \quad \det(A_2 x_1 + A_1 x_2) = 0$$

is a defining binary cubic form for C' in $\mathbf{P}(\mathcal{E})$. Indeed, this follows by noting that 4 points in \mathbf{P}^2 in general position define a pencil of quadrics, and that the 3 dual points are in a natural bijection with the degenerate members of this pencil. From expression (27) one explicitly checks that the scrollar invariants of φ' are b_1, b_2 , e.g., using [14, Thm.9.1]. One can also use (27) to verify that C' is indeed the geometric counterpart of Lagrange’s cubic resolvent, using the formulas from [3, p. 1351].

(6.4). Next, we consider the subgroup

$$\mathrm{AGL}_1(\mathbf{F}_5) = \langle (1\ 2\ 3\ 4\ 5), (1\ 2\ 4\ 3) \rangle \subseteq S_5.$$

In Galois theory, the resolvent of a quintic polynomial $f(x)$ with respect to $\mathrm{AGL}_1(\mathbf{F}_5)$ is known as “Cayley’s sextic resolvent”; its main use lies in determining whether the equation $f(x) = 0$ is solvable by radicals [20, Cor.13.2.11]. When applied to a simply branched degree 5 cover of \mathbf{P}^1 by a curve of genus g , we obtain a degree 6 cover of \mathbf{P}^1 by a curve of genus $3g + 7$, with the following scrollar invariants:

Corollary 37. *Consider a simply branched degree 5 cover $\varphi : C \rightarrow \mathbf{P}^1$. The degree 6 cover $\mathrm{R}_{\mathrm{AGL}_1(\mathbf{F}_5)} \varphi : \mathrm{R}_{\mathrm{AGL}_1(\mathbf{F}_5)} C \rightarrow \mathbf{P}^1$ has scrollar invariants*

$$\{b_1^{(2)}, b_2^{(2)}, b_3^{(2)}, b_4^{(2)}, b_5^{(2)}\},$$

where the $b_j^{(2)}$'s denote the elements of the splitting type of the second syzygy bundle in the relative minimal resolution of φ .

Proof. Use $\text{Ind}_{\text{AGL}_1(\mathbb{F}_5)}^{S_5} \mathbf{1} \cong V_{(5)} \oplus V_{(2^2,1)}$ in combination with Theorems 3 and 4. \square

Cayley's sextic resolvent also appears in Bhargava's work on quintic ring parametrizations [4], and a proof of Corollary 37 can also be deduced from that work. The details of this approach can be found in the master thesis of the second-listed author [54, Thm.4.13], where this was studied in the context of lifting pentagonal curves from finite fields to characteristic zero, for use in Tuitman's point counting algorithm [15, §5]. As in the case of Recillas' trigonal construction, this approach in fact allows for a relaxation of the simple branching assumption.

(6.5). From degree 6 onward, there exist resolvents whose scrollar invariants we can no longer relate to known data of $\varphi : C \rightarrow \mathbf{P}^1$. For $d = 6$, these scrollar invariants arise from the partitions (2^3) and (3^2) ; note that these partitions are dual to each other. Since $\dim V_{(2^3)} = 5$, the scrollar invariants of (2^3) are certain integers $1 \leq a_1 \leq \dots \leq a_5$, which sum up to $3g + 15$ by Proposition 29. By Proposition 30 the scrollar invariants of (3^2) are $g + 5 - a_1, \dots, g + 5 - a_5$, which sum up to $2g + 10$. We call a_1, a_2, \dots, a_5 the "exotic invariants" of C with respect to φ . The terminology comes from the exotic embedding $S_5 \hookrightarrow S_6$ realizing S_5 as a transitive subgroup of S_6 ; we denote this subgroup by S'_5 . It is unique up to conjugation; one realization is $\langle (1\ 2\ 3\ 4), (1\ 5\ 6\ 2) \rangle$. With respect to S'_5 , the resolvent of a simply branched degree 6 cover of \mathbf{P}^1 by a curve of genus g is another degree 6 cover of \mathbf{P}^1 , now by a curve of genus $3g + 10$.

Corollary 38. *Consider a simply branched degree 6 cover $\varphi : C \rightarrow \mathbf{P}^1$. The scrollar invariants of the degree 6 cover $R_{S'_5} C$ with respect to $R_{S'_5} \varphi$ are given by the exotic invariants of C with respect to φ .*

Proof. This follows from $\text{Ind}_{S'_5}^{S_6} \mathbf{1} \cong V_{(6)} \oplus V_{(2^3)}$ along with Theorem 4. \square

We wonder whether a deeper understanding of these exotic invariants could be key towards a better understanding of the Hurwitz spaces $\mathcal{H}_{6,g}$, where the most pressing question is whether they are unirational or not [46, §1]. Here are two concrete first problems:

Problem 39. *Find an alternative interpretation for the exotic invariants a_i , directly in terms of the morphism $\varphi : C \rightarrow \mathbf{P}^1$.*

Problem 40. *Find simply branched degree 6 morphisms $C \rightarrow \mathbf{P}^1$ and $C' \rightarrow \mathbf{P}^1$ having the same scrollar invariants, the same Schreyer invariants, but different exotic invariants.*

(6.6). **Some further infinite families.** In arbitrary degree $d \geq 2$, we have already discussed the resolvent with respect to A_{d-1} in Corollary 31, as well as the resolvent with respect to $S_2 \times S_{d-2}$ in Theorem 5. Let us extend this list somewhat further:

Corollary 41. *Consider a simply branched cover $\varphi : C \rightarrow \mathbf{P}^1$ of degree $d \geq 4$. Let $\{e_1, e_2, \dots, e_{d-1}\}$ be its scrollar invariants and let $\{b_1, b_2, \dots, b_{d(d-1)/3}\}$ be the splitting type of the first syzygy bundle of its relative minimal resolution. Then:*

- the scrollar invariants of $R_{S_{d-2}} C$ with respect to $R_{S_{d-2}} \varphi$ are

$$\{e_i\}_i \cup \{e_i\}_i \cup \{e_i + e_j\}_{i < j} \cup \{b_i\}_i,$$
- the scrollar invariants of $R_{S_2 \times A_{d-2}} C$ with respect to $R_{S_2 \times A_{d-2}} \varphi$ are

$$\{e_i\}_i \cup \{b_i\}_i \cup \{g + d - 1 - e_i\}_i \cup \{g + d - 1 - e_i - e_j\}_{i < j},$$
- the scrollar invariants of $R_{A_{d-2}} C$ with respect to $R_{A_{d-2}} \varphi$ are

$$\begin{aligned} & \{e_i\}_i \cup \{e_i\}_i \cup \{e_i + e_j\}_{i < j} \cup \{b_i\}_i \\ & \cup \{g + d - 1 - e_i\}_i \cup \{g + d - 1 - e_i\}_i \\ & \cup \{g + d - 1 - e_i - e_j\}_{i < j} \cup \{g + d - 1 - b_i\}_i \cup \{g + d - 1\}, \end{aligned}$$

where the unions are as multi-sets and where g denotes the genus of C .

Proof. This follows from the decompositions¹¹

$$\begin{aligned} \text{Ind}_{S_{d-2}}^{S_d} \mathbf{1} &\cong V_{(d)} \oplus V_{(d-1,1)}^2 \oplus V_{(d-2,1^2)} \oplus V_{(d-2,2)}, \\ \text{Ind}_{S_2 \times A_{d-2}}^{S_d} \mathbf{1} &\cong V_{(d)} \oplus V_{(d-1,1)} \oplus V_{(d-2,2)} \oplus V_{(3,1^{d-3})} \oplus V_{(2,1^{d-2})}, \\ \text{Ind}_{A_{d-2}}^{S_d} \mathbf{1} &\cong V_{(d)} \oplus V_{(d-1,1)}^2 \oplus V_{(d-2,1^2)} \oplus V_{(d-2,2)} \\ &\quad \oplus V_{(3,1^{d-3})} \oplus V_{(2,1^{d-2})}^2 \oplus V_{(2^2,1^{d-4})} \oplus V_{(1^d)} \end{aligned}$$

along with Proposition 2 and Theorems 3 and 4; in the case of A_{d-2} we also used duality. \square

(6.7). For the sake of illustration, we also include an infinite family of resolvents whose scrollar invariants we cannot express purely in terms of $\{e_i\}_i$ and $\{b_i\}_i$ (unless these would turn out to be related to the scrollar invariants of $(d-3, 3)$; this is related to Problem 40).

Corollary 42. *Consider a simply branched cover $\varphi : C \rightarrow \mathbf{P}^1$ of degree $d \geq 6$. Let $\{e_1, e_2, \dots, e_{d-1}\}$ be its scrollar invariants, let*

$$\{b_1^{(1)}, b_2^{(1)}, \dots, b_{d(d-1)/3}^{(1)}\}, \quad \text{resp.} \quad \{b_1^{(2)}, b_2^{(2)}, \dots, b_{d(d-1)/3}^{(2)}\},$$

¹¹Note that, for $d = 4, 5$, some terms may coincide. E.g., for $d = 4$ the terms $V_{(d-1,1)}$ and $V_{(3,1^{d-3})}$ are the same, and then so are the corresponding multi-sets of scrollar invariants $\{e_i\}_i$ and $\{g + d - 1 - e_i - e_j\}_{i < j}$.

be the splitting types of the first, resp. second, syzygy bundle of its relative minimal resolution, and let $\{c_1, c_2, \dots, c_{d(d-1)(d-5)/6}\}$ be the scrollar invariants of the partition $(d-3, 3)$ with respect to φ . The scrollar invariants of $\mathbb{R}_{S_2 \times S_{d-3}} C$ with respect to $\mathbb{R}_{S_2 \times S_{d-3}} \varphi$ are given by

$$\{e_i\}_i \cup \{e_i\}_i \cup \{e_i + e_j\}_{i < j} \cup \{b_i^{(1)}\}_i \cup \{b_i^{(1)}\}_i \cup \{b_i^{(2)}\}_i \cup \{c_i\}_i,$$

as a union of multi-sets.

Proof. This follows from

$$\text{Ind}_{S_2 \times S_{d-3}}^{S_d} \mathbf{1} \cong V_{(d)} \oplus V_{(d-1,1)}^2 \oplus V_{(d-2,1^2)} \oplus V_{(d-2,2)}^2 \oplus V_{(d-3,2,1)} \oplus V_{(d-3,3)}$$

together with Proposition 2, Theorem 3 and Theorem 4. The fact that $(d-3, 3)$ comes equipped with $d(d-1)(d-5)/6$ scrollar invariants follows from the hook length formula. \square

Note that the invariants c_i sum up to $(d-2)(d-5)(g+d-1)/2$ in view of Proposition 29. If $d=6$ then these invariants are dual to the exotic invariants discussed in (6.5).

(6.8). **Curves on Hirzebruch surfaces.** Consider a smooth curve C on the Hirzebruch surface $F_e = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(e))$ of invariant $e \geq 0$, along with the morphism $\varphi : C \rightarrow \mathbf{P}^1$ induced by the bundle map $\pi : F_e \rightarrow \mathbf{P}^1$. Assume that this morphism is dominant of degree $d \geq 2$, and simply branched. We will give a conjectural description of the scrollar invariants of each partition of d with respect to φ in terms of the “bidegree” of C , by which we mean the tuple $(c, d) \in \mathbf{Z}^2$ such that $C \sim dE + (c+de)F$, with F a fiber of π and E a section with self-intersection $-e$. Equivalently, one can view F_e as the projective toric surface polarized by the lattice polygon $\Delta_{c,d,e}$ from Figure 3, and then C is defined by a sufficiently general bivariate polynomial that is

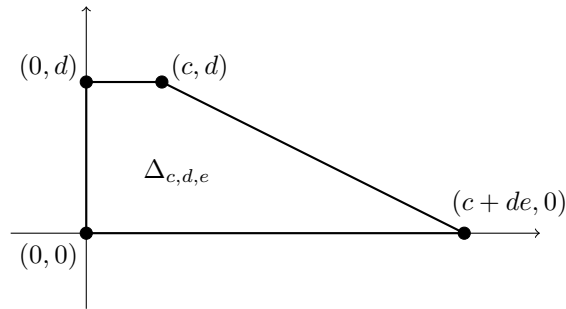


FIGURE 3. The lattice polygon corresponding to bidegree (c, d) on F_e .

supported on this polygon (with φ corresponding to projection on the first coordinate). The most interesting special cases are $e=0$, in which case we are looking at smooth curves on $\mathbf{P}^1 \times \mathbf{P}^1$ of bidegree (c, d) in the traditional sense, and $e=c=1$, corresponding to smooth plane curves of degree $d+1$.

The scrollar invariants of C with respect to φ are given by

$$e_i = c + ie, \quad i = 1, \dots, d - 1.$$

This follows from [14, Thm.9.1], since e_i equals the length of the longest line segment having lattice end points in the interior of $\Delta_{c,d,e}$ at height $d - i$. Consequently, the genus of C equals $(d - 1)(c + de/2 - 1)$. This is the number of lattice points in the interior of $\Delta_{c,d,e}$.

(6.9). Let λ be a partition of d , and consider the following recipe for attaching an integer $e(T)$ to a standard Young tableau T of shape λ . Start by writing down the “reading word” of T , which is obtained by concatenating its rows, starting from the bottom row. For example, the reading word of the tableau

1	3	5	7
2	4		
6			

of shape $(4, 2, 1)$ is $6\,2\,4\,1\,3\,5\,7$. Consider the set $I(T)$ of indices $i \in \{1, \dots, d - 1\}$ for which $i + 1$ appears to the left of i in this reading word. Then we let

$$e(T) = \sum_{i \in I(T)} e_i$$

which for the above example equals $e_1 + e_3 + e_5 = c + 9e$. We propose:

Conjecture 43. *Let $e \geq 0$ be an integer and let C be a smooth curve on F_e . Let $C \sim dE + (c + de)F$ for integers $d \geq 2$, $c \geq 0$. Assume that the map $\varphi : C \rightarrow \mathbf{P}^1$ induced by the bundle map $F_e \rightarrow \mathbf{P}^1$ is simply branched. Then the multi-set of scrollar invariants of any partition $\lambda \vdash d$ with respect to φ is given by*

$$\{e(T) \mid T \text{ is a Young tableau of shape } \lambda\}.$$

Note that this multi-set is typically not consecutive, let alone balanced. Of course, this does not contradict [35, Ex.1.3.7] since smooth curves on Hirzebruch surfaces have a very small locus inside Hurwitz space.

(6.10). For the partition $(d - 1, 1)$ the conjecture just returns the scrollar invariants $\{e_1, e_2, \dots, e_{d-1}\}$ of C with respect to φ , as wanted. By Proposition 2 the conjecture is also true for hooks. It also holds for the partition $(d - 2, 2)$ thanks to Theorem 3 and [13, Thm.16]; by duality, the case $(2^2, 1^{d-4})$ is covered as well.

Thus the first open cases appear in degree $d = 6$. We have carried out several experiments in Magma [9]. Each experiment amounted to computing well-chosen resolvents of some random curve of given bidegree over a large

finite field, and recovering the scrollar invariants of these resolvent curves using the command `ShortBasis()`; our code is available at <https://homes.esat.kuleuven.be/~wcastry>. Each time the output matched with the prediction from Conjecture 43, when combined with Theorem 4. Our choices covered the three remaining partitions (2^3) , $(3, 2, 1)$, (3^2) of $d = 6$, as well as some new partitions of $d = 7, 8$.

Let us emphasize that Conjecture 43 is more than just a guess interpolating between all known cases: these numerics naturally show up when studying scrollar invariants of the S_d -closure of monogenic extensions [5, §6], and this is how we came up with Conjecture 43 in the first place. In fact, we believe that the direct analogue of our conjecture applies to curves in a much more general class of toric surfaces than Hirzebruch surfaces; however, for arbitrary toric surfaces the combinatorics becomes more subtle and we expect the need for certain correction terms, as is already apparent from [13].

7. APPLICATIONS AND CONCLUDING REMARKS

(7.1). **Curves with highly non-balanced scrollar invariants.** Using our results, many new multi-sets of integers can be shown to be realizable as the multi-set of scrollar invariants of some \mathbf{P}^1 -cover. Here is an example statement:

Proposition 44. *Consider integers $d \geq 2$ and $g \geq d - 1$, and let k be an algebraically closed field with $\text{char } k = 0$ or $\text{char } k > d$. Let $e_1 \leq \dots \leq e_{d-1}$ be integers summing to $g + d - 1$ such that $e_{d-1} - e_1 \leq 1$. There exists a genus g curve C over k along with a morphism $\varphi : C \rightarrow \mathbf{P}^1$ whose multi-set of scrollar invariants is given by*

$$\{e_1, e_2, \dots, e_{d-1}, g + d - 1 - e_1, g + d - 1 - e_2, \dots, g + d - 1 - e_{d-1}, g + d - 1\}.$$

Proof. Let $\varphi' : C' \rightarrow \mathbf{P}^1$ be a general element of $\mathcal{H}_{d,g}$. It is simply branched, and by a result of Ballico [1] its multi-set of scrollar invariants is balanced, i.e., it is given by $\{e_1, e_2, \dots, e_{d-1}\}$. Then take φ to be $R_{A_{d-1}} \varphi' : R_{A_{d-1}} C' \rightarrow \mathbf{P}^1$ and apply Corollary 31. \square

Similar results can be obtained using the resolvents with respect to A_{d-2} , S_{d-2} , $S_2 \times A_{d-2}$ and $S_2 \times S_{d-2}$, whose scrollar invariants were determined in Corollary 41 and Theorem 5, using generic balancedness of the first syzygy bundle [10, Main Thm.] in addition to Ballico's result.

(7.2). The typical resolvent curve is expected to have highly non-balanced scrollar invariants. Indeed, by Theorem 4 the multi-set of scrollar invariants of the resolvent with respect to some subgroup $H \subseteq S_d$ is naturally subdivided into subsets, one for every partition λ appearing in $\text{Ind}_H^{S_d} \mathbf{1}$. From

¹²Defining equations for these resolvents are found using a fast t -adic method that was suggested to us by Frederik Vercauteren and which may be found interesting in its own right.

Landesman–Litt [35, Ex. 1.3.7] we know that, generically, the scrollar invariants corresponding to λ are all contained in the interval

$$\left[\frac{\text{vol}_K(\lambda)}{\dim V_\lambda} - \frac{\dim V_\lambda - 1}{2}, \frac{\text{vol}_K(\lambda)}{\dim V_\lambda} + \frac{\dim V_\lambda - 1}{2} \right].$$

These values live in regimes that vary strongly with λ . Thus, statements like Proposition 44 contrast with previously known ways of constructing curves with prescribed scrollar invariants, such as [14, 17, 22], which produce instances that are close to being balanced.

(7.3). **New bounds on Schreyer invariants.** The existing literature reports on several bounds on scrollar invariants. The most important such bound is the Maroni bound, but see e.g. [21, Prop.2.6], [40, Cond.1–3] and [53] for other examples. These results can be combined with Theorem 3 for obtaining bounds on the Schreyer invariants of a simply branched cover.

For instance, applying the Maroni bound to a resolvent cover gives the following:

Lemma 45. *Let $\varphi : C \rightarrow \mathbf{P}^1$ be a simply branched cover of degree $d \geq 2$ over a field k with $\text{char } k = 0$ or $\text{char } k > d$. Let H be a proper subgroup of S_d and let $\lambda \vdash d$ be such that V_λ appears in $\text{Ind}_H^{S_d} \mathbf{1}$. Then the scrollar invariants $e_{\lambda,j}$ of λ with respect to φ satisfy*

$$e_{\lambda,j} \leq \frac{|\{\text{transpositions } \sigma \notin H\}|}{\binom{d}{2}} (g + d - 1),$$

where g denotes the genus of C .

Proof. By Theorem 4 the scrollar invariants of λ appear among those of $R_H C$ with respect to $R_H \varphi$. The Maroni bound for $R_H C$ yields that

$$e_{\lambda,j} \leq \frac{2g(R_H C) + 2[S_d : H] - 2}{[S_d : H]}.$$

The genus formula from Theorem 27 then gives the stated result. \square

Applying this lemma to a well-chosen subgroup of S_d , we get the following bounds on the scrollar and Schreyer invariants. This gives some general range in which the scrollar invariants of a partition λ can live. However, determining the actual range of possibilities seems very difficult, as this is not even known for the usual scrollar invariants e_1, \dots, e_{d-1} .

Theorem 46. *Let $\varphi : C \rightarrow \mathbf{P}^1$ be a simply branched cover of degree $d \geq 4$ over a field k with $\text{char } k = 0$ or $\text{char } k > d$. Let $\lambda = (d_1, \dots, d_r)$ be a partition of d . Then the scrollar invariants $e_{\lambda,j}$ of λ with respect to φ satisfy*

$$e_{\lambda,j} \leq \frac{d^2 - \sum_i d_i^2}{d(d-1)} (g + d - 1).$$

Proof. By [44, Cor. 2.4.7], the representation V_λ appears in $\text{Ind}_{S_\lambda}^{S_d} \mathbf{1}$. Hence we may apply Lemma 45 with $H = S_\lambda$. The result then follows from the fact that the number of transpositions in S_λ is equal to

$$\binom{d_1}{2} + \binom{d_2}{2} + \dots + \binom{d_r}{2}. \quad \square$$

Corollary 47. *For $i \in \{1, 2, \dots, d-3\}$, the elements $b_j^{(i)}$ of the splitting type of the i th syzygy bundle in the relative minimal resolution of C with respect to φ are contained in*

$$\left[\frac{i(i+1)+2}{d(d-1)}(g+d-1), \frac{(i+1)(2d-i-2)-2}{d(d-1)}(g+d-1) \right].$$

In particular, all $b_j^{(i)}$ are non-negative.

Proof. For the upper bound, we apply the previous theorem to λ_i for $i = 2, \dots, d-2$ in combination with Theorem 3. For the lower bound, we use the duality of the Schreyer invariants discussed in (5.6). \square

We believe that this result remains valid under weaker conditions than simple branching.

(7.4). The proof of Corollary 47 used the resolvent with respect to $S_2 \times S_{d-i-1}$. In general however, it is unclear which resolvent gives the best upper bound. E.g., using the maximal resolvent (i.e., the Galois closure) would only give an upper bound of the form

$$b_j^{(i)} \lesssim g,$$

which is always worse than the bound given here. To obtain a good bound on the scrollar invariants $e_{\lambda,i}$, one wants a subgroup $H \subseteq S_d$ such that V_λ appears in $\text{Ind}_H^{S_d} \mathbf{1}$ and such that H contains as many transpositions as possible, in view of Lemma 45. For the partitions λ_i , the reader can check that the optimal Young subgroups are indeed $S_2 \times S_{d-i}$. One can also apply Lemma 45 to non-Young subgroups to obtain bounds on the scrollar invariants. However, it seems that Young subgroups always give the strongest possible bounds.

(7.5). If C is a general curve in the Hurwitz space $\mathcal{H}_{d,g}$ then we can give stronger upper bounds on the scrollar invariants of C .

Theorem 48. *Let $\varphi : C \rightarrow \mathbf{P}^1$ be a general element in the Hurwitz space $\mathcal{H}_{d,g}$ and let λ be a partition of d . Let i be the number of boxes outside the first row of the Young diagram of λ . Then the scrollar invariants $e_{\lambda,j}$ of λ with respect to φ satisfy*

$$e_{\lambda,j} \leq \frac{i}{d-1}g + 2i.$$

In particular, if $d \geq 4$ then the splitting types $b_j^{(i)}$ satisfy

$$b_j^{(i)} \in \left[\frac{i}{d-1}g + 2(i+1) - d - 1, \frac{i+1}{d-1}g + 2(i+1) \right]$$

for $i = 1, \dots, d-3$.

Proof. Let L be the Galois closure of the function field $k(C)/k(t)$ and recall that we denote by W_λ the isotypic component. An adaptation of the proof of Lemma 20 shows that $W_\lambda \subset W_{(d-1,1)}^i$. So if $\alpha_1, \dots, \alpha_{d-1}$ is a reduced basis for $V_1 = W_{(d-1,1)} \cap L^{S_{d-1}}$, then W_λ has a basis consisting of elements which are $k[t]$ -linear combinations of conjugates of elements of the form

$$\alpha_{j_1}^{(m_1)} \cdots \alpha_{j_i}^{(m_i)}.$$

Note that such elements are integral over $k[t]$ and hence the scrollar invariants of λ satisfy

$$e_{\lambda,j} \leq i \max_{\ell} e_\ell,$$

where e_1, \dots, e_{d-1} are the usual scrollar invariants of $C \rightarrow \mathbf{P}^1$. The condition that C is general implies that the e_ℓ are balanced, so $e_\ell \leq \frac{g}{d-1} + 2$ for every ℓ . This gives the desired statement.

The upper bound on the splitting types $b_j^{(i)}$ follows from Theorem 3, while the lower bound follows from duality as in Section (5.6). \square

(7.6). **Gassmann equivalence implies scrollar equivalence.** Two subgroups H_1, H_2 of a finite group G are called ‘‘Gassmann equivalent’’ if for every conjugacy class \mathcal{C} of G it holds that $|H_1 \cap \mathcal{C}| = |H_2 \cap \mathcal{C}|$. For example, the subgroups $H_1 = \langle (12)(34), (13)(24) \rangle$ and $H_2 = \langle (12)(34), (12)(56) \rangle$ of S_6 are Gassmann equivalent, despite the fact that they are not conjugate. A classical result by Gassmann states that if L is a finite Galois extension of \mathbf{Q} with Galois group G , then H_1, H_2 are Gassmann equivalent if and only if L^{H_1} and L^{H_2} are ‘‘arithmetically equivalent’’, i.e., they have the same Dedekind zeta function [48, Thm. 1.23]. In that case, they necessarily have the same degree and discriminant.

This story partly breaks down in the function field setting, see [48, §3.1.1] for a discussion. However, without much effort we can conclude:

Theorem 49. *Let $\varphi : C \rightarrow \mathbf{P}^1$ be a simply branched degree d cover, and let H_1, H_2 be two Gassmann equivalent subgroups of S_d . Then the resolvent covers $R_{H_1} \varphi : R_{H_1} C \rightarrow \mathbf{P}^1$ and $R_{H_2} \varphi : R_{H_2} C \rightarrow \mathbf{P}^1$ have the same degree, genus, and scrollar invariants.*

Proof. $\text{Ind}_{H_1}^{S_d} \mathbf{1} \cong \text{Ind}_{H_2}^{S_d} \mathbf{1}$ by [48, §1.4.2], so the claim about the scrollar invariants follows from Theorem 4. This implies the claims about degree and genus. \square

Equality of degree and genus has a clear counterpart for number fields (degree resp. discriminant), while the scrollar invariants of a curve correspond

to the successive minima of the Minkowski lattice of a number field. An analogue of Theorem 49 in the number field setting was recently proved by the second-listed author [55].

(7.7). **On the simple branching assumption.** Our first crucial use of the simple branching assumption was in the proof of Theorem 27, giving a formula for the genus of a resolvent curve. We recall that this genus formula was used to prove our volume formula from Proposition 29, therefore the assumption is an important ingredient in the proof of Theorem 3. However, by using a more careful notion of “resolvent cover”, we believe that it should be possible to get rid of the simple branching assumption. On the geometry side, this begs for a vast generalization of Recillas’ trigonal construction. Here, an interesting first problem is to reinterpret some existing generalizations, e.g., as discussed in [23, §4.4] and [33], in terms of Galois theory. On the algebraic side, we seem to require a theory of resolvents on the level of rings, rather than fields, which is reminiscent of Lagrange’s classical theory of resolvent polynomials [20, §12.1]. For degrees $d = 4$ and $d = 5$ it should be possible to extract such a theory from Bhargava’s work on ring parametrizations [3, 4], see also [57]. In the general case, the key ingredient seems to be Bhargava and Satriano’s notion of “ S_d -closure” [5]. In particular, we still expect the relevant representation theory to be that of S_d , even in the case of a smaller Galois closure.

In some more detail, if $\varphi : C \rightarrow \mathbf{P}^1$ is an S_d -cover with arbitrary ramification, then for certain subgroups $H \subseteq S_d$ the genus of $R_H C$ might be smaller than what is predicted by Theorem 27. The proof of Theorem 3 suggests that, in order to fix this, one should allow for the use of non-maximal orders in L^H , i.e., singular models of $R_H C$. This is also apparent from Recillas’ trigonal construction applied to arbitrary S_4 -covers of degree 4, which produces singular curves in the presence of ramification of type (2^2) or (4) . More generally, the trigonal construction suggests that if φ is a G -cover for some arbitrary $G \subseteq S_d$, then the “correct” resolvent curve with respect to $H \subseteq S_d$ may be singular and/or reducible. Algebraically, the corresponding resolvent cover is described in terms of orders in an étale algebra that may not be a field. However, the notions of reduced bases and scrollar invariants naturally carry over to this setting, and we expect that all our main results continue to hold for this generalized notion of scrollar invariants.

(7.8). Even in the case of non-simple branching, it might still happen that this more carefully constructed resolvent curve is smooth and geometrically integral, in which case statements like in Section 6 should continue to hold without modification. For example, both Casnati’s result [11, Def. 6.3-6.4] and Proposition 35 are illustrations of this phenomenon. However, we expect that most subgroups $H \subseteq S_d$ tolerate a small number of ramification patterns only. Furthermore, we expect that this is again determined by representation theory. In more detail, for every partition $\lambda \vdash d$ there should be a list of “good ramification” patterns, and then the resolvent associated to

H will be smooth and geometrically integral if and only if there is good ramification for every λ appearing in $\text{Ind}_H^{S_d} \mathbf{1}$. For instance, simple branching is good for all partitions of d , and all ramification patterns should be good for the partition $(d - 1, 1)$. For $\lambda = (d - 2, 2)$ we expect that also $(3, 1^{d-3})$ is good; this is true for Recillas’ trigonal construction in case $d = 4$, while for arbitrary $d \geq 4$ this follows from Lemma 32 under the assumption that φ is an S_d -cover.

(7.9). If we indeed manage to get rid of the simple branching assumption, then the resulting generalization of Theorem 3 will provide us with an alternative, syzygy-free way of defining the Schreyer invariants of any cover $\varphi : C \rightarrow \mathbf{P}^1$. We remark that this definition would have a natural counterpart for number fields, in terms of successive minima of sublattices of the S_d -closure of their ring of integers.

(7.10). **Counting function fields.** We end by noting that this project started with a rediscovery of Casnati’s result during an investigation of the secondary term in the counting function for quartic extensions of $\mathbf{F}_q(t)$ having bounded discriminant, where \mathbf{F}_q denotes a finite field whose cardinality q satisfies $\gcd(q, 6) = 1$. This study was motivated by the Ph.D. work of the third-listed author [59] who determined the secondary term in the cubic case, thereby settling the $\mathbf{F}_q(t)$ -counterpart of Roberts’ conjecture [43],¹³ which reads that

$$N_3(X) = \underbrace{\frac{1}{3\zeta(3)}}_{=0.277\dots} X + \underbrace{\frac{4(1 + \sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}}_{=-0.403\dots} X^{5/6} + o(X^{5/6}),$$

where X is a real parameter tending to infinity and $N_3(X)$ denotes the number of non-isomorphic cubic extensions $K \supseteq \mathbf{Q}$ for which $|\Delta_K| \leq X$. Proving the $\mathbf{F}_q(t)$ -analogue of this statement essentially boils down to estimating the number of non-isomorphic degree 3 covers of \mathbf{P}^1 over \mathbf{F}_q by curves of a given genus g , which can be done in bulks by first enumerating all possibilities for the scrollar invariants e_1, e_2 . As it turns out, the appearance of a negative term in $X^{5/6}$ is naturally related to the offset in this enumeration coming from the Maroni bound $e_2 \leq (2g + 4)/3$.

A heuristic reasoning à la Roberts makes it reasonable¹⁴ to expect a similar negative term of order $X^{5/6}$ in the counting function $N_4(X)$ for quartic number fields K with $|\Delta_K| \leq X$. However, it seems a hard open problem to make this prediction precise, and valuable support in its favour would be lended by a proof of its $\mathbf{F}_q(t)$ -analogue, which essentially amounts to estimating the number of non-isomorphic degree 4 covers of \mathbf{P}^1 by curves with a given genus g . If this can be done by mimicking the ideas of [59],

¹³Now a theorem thanks to independent work of Bhargava–Shankar–Tsimmerman [7] and Taniguchi–Thorne [49].

¹⁴This is based on Yukie’s analysis [58] of the quartic Shintani zeta function, as was explained to us by Takashi Taniguchi in personal communication.

it will involve several technical sieving steps, dealing with covers that are not necessarily simply branched, so this lies beyond the scope of this article. However, it is possible to make a rough back-of-the-envelope analysis suggesting that the exponent $5/6$ is again naturally related to a bound of Maroni type, but now on Schreyer's invariants b_1, b_2 rather than on the scrollar invariants e_1, e_2, e_3 . And, of course, the best reason for the existence of such a bound is that Schreyer's invariants *are* scrollar invariants, by Theorem 3.

REFERENCES

1. E. Ballico, *A remark on linear series on general k -gonal curves*, Bolletino dell'Unione Matematica Italiana **7** (1989), 195–197.
2. K. Behnke, *On projective resolutions of Frobenius algebras and Gorenstein rings*, Mathematische Annalen **257** (1981), 219–238.
3. M. Bhargava, *Higher composition laws III: The parametrization of quartic rings*, Annals of Mathematics **159** (2004), no. 3, 1329–1360.
4. ———, *Higher composition laws IV: The parametrization of quintic rings*, Annals of Mathematics **167** (2008), no. 1, 53–94.
5. M. Bhargava and M. Satriano, *On a notion of “Galois closure” for extensions of rings*, Journal of the European Mathematical Society **16** (2014), no. 9, 1881–1913.
6. M. Bhargava, A. Shankar, T. Taniguchi, F. Thorne, J. Tsimerman, and Y. Zhao, *Bounds on 2-torsion in class groups of number fields and integral points on elliptic curves*, Journal of the American Mathematical Society **33** (2020), no. 4, 1087–1099.
7. M. Bhargava, A. Shankar, and J. Tsimerman, *On the Davenport-Heilbronn theorems and second order terms*, arXiv:1005.0672 [math] (2012), arXiv: 1005.0672.
8. C. Bopp and M. Hoff, *Resolutions of general canonical curves on rational normal scrolls*, Archiv der Mathematik **105** (2015), no. 3, 239–249.
9. W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, Journal of Symbolic Computation **24** (1997), no. 3–4, 235–265.
10. G. Bujokas and A. Patel, *Invariants of a general branched cover of \mathbf{P}^1* , International Mathematics Research Notices **2021** (2021), no. 6, 4564–4604.
11. G. Casnati, *Covers of algebraic varieties III. The discriminant of a cover of degree 4 and the trigonal construction*, Transactions of the American Mathematical Society **350** (1998), 1359–1378.
12. G. Casnati and T. Ekedahl, *Covers of algebraic varieties I. A general structure theorem, covers of degree 3, 4 and Enriques surfaces*, Journal of Algebraic Geometry **5** (1995), 439–460.
13. W. Castryck and F. Cools, *Intrinsicness of the Newton polygon for smooth curves on $\mathbf{P}^1 \times \mathbf{P}^1$* , Revista Matemática Complutense **30** (2017), no. 2, 233–258.
14. ———, *Linear pencils encoded in the Newton polygon*, International Mathematics Research Notices **2017** (2017), 2998–3049.
15. W. Castryck and F. Vermeulen, *Lifting low-gonal curves for use in Tuitman's algorithm*, Proceedings of ANTS-XIV. MSP Open Book Series **4** (2020), 109–125.
16. W. Castryck, F. Vermeulen, and Zhao Y., *Scrollar invariants, syzygies and representations of the symmetric group*, Journal für die reine und angewandte Mathematik, to appear.
17. M. Coppens, *Existence of pencils with prescribed scrollar invariants of some general type*, Osaka Journal of Mathematics **36** (1999), no. 4, 1049–1057.
18. M. Coppens, C. Keem, and G. Martens, *Primitive linear series on curves*, Manuscripta Mathematica **77** (1992), 237–264.
19. M. Coppens and G. Martens, *Linear series on a general k -gonal curve*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **69** (1999), 347–371.

20. D. Cox, *Galois Theory*, John Wiley & Sons, 2012.
21. A. Deopurkar and A. Patel, *The Picard rank conjecture for the Hurwitz spaces of degree up to five*, *Algebra & Number Theory* **9** (2015), no. 2, 459–492.
22. ———, *Vector bundles and finite covers*, arXiv preprint (2019), <https://arxiv.org/abs/1608.01711>.
23. R. Donagi, *The Schottky problem*, *Proceedings of Theory of Moduli*, *Lecture Notes in Mathematics* **1337** (1985), 84–137.
24. D. Eisenbud, N. Elkies, J. Harris, and R. Speiser, *On the Hurwitz scheme and its monodromy*, *Compositio Mathematica* **77** (1991), no. 1, 95–117.
25. D. Eisenbud and J. Harris, *On varieties of minimal degree (a centennial account)*, *Proceedings of Symposia in Pure Mathematics* **46** (1987), 3–13.
26. W. Fulton, *Hurwitz schemes and irreducibility of moduli of algebraic curves*, *Annals of Mathematics* **90** (1969), no. 3, 542–575.
27. W. Fulton and J. Harris, *Representation theory: a first course*, *Graduate Texts in Mathematics*, Springer, 1991.
28. W. T. Gan, B. Gross, and G. Savin, *Fourier coefficients of modular forms on G_2* , *Duke Mathematical Journal* **115** (2002), no. 1, 105–169.
29. M. Hamermesh, *Group theory and its application to physical problems*, *Dover Books on Physics and Chemistry*, Dover Publications, 1962.
30. F. Hess, *Computing Riemann-Roch spaces in algebraic function fields and related topics*, *Journal of Symbolic Computation* **33** (2002), no. 4, 425–445.
31. ———, *Algorithmics of function fields*, *Commented slides for the UNCG Summer School in Computational Number Theory: Function Fields*, 2016, <https://mathstats.uncg.edu/sites/number-theory/summerschool/2016/uncg-hess-comments.pdf>, accessed 21/09/22.
32. F. Hivert and N. M. Thiéry, *Symmetric functions and the rational Steenrod algebra*, *CRM Proceedings and Lecture Notes* **35** (2002), 91–125.
33. M. Hoff and U. Mayer, *The osculating cone to special Brill–Noether loci*, *Collectanea Mathematica* **66** (2015), no. 3, 387–403.
34. A. Landesman and D. Litt, *Applications of the algebraic geometry of the Putman–Wieland conjecture*, arXiv preprint (2022), <https://arxiv.org/pdf/2209.00718>, v2.
35. ———, *Geometric local systems on very general curves and isomonodromy*, arXiv preprint (2022), <https://arxiv.org/abs/2202.00039>, v2.
36. A. Landesman, R. Vakil, and M. M. Wood, *Low degree Hurwitz stacks in the Grothendieck ring*, arXiv preprint (2022), <https://arxiv.org/pdf/2203.01840>, v1.
37. H. W. Lenstra, J. Pila, and C. Pomerance, *A hyperelliptic smoothness test, II*, *Proceedings of the London Mathematical Society* **84** (2002), no. 1, 105–146.
38. V.B. Mehta and C.S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, *Mathematische Annalen* **248** (1980), no. 3, 205–239.
39. J. Neukirch, *Algebraic number theory*, Springer-Verlag Berlin, 1999.
40. A. Patel, *The geometry of Hurwitz space*, Ph.D. thesis, 2013.
41. C. Peikert and A. Rosen, *Lattices that admit logarithmic worst-case to average-case connection factors*, full version, available at <https://eprint.iacr.org/2006/444>, of eponymous paper published at *Proceedings of the 39th ACM Symposium on Theory of Computing* (2007), 478–487.
42. S. Recillas, *Jacobians of curves with g_4^1 's are the Pryms of trigonal curves*, *Boletín de la Sociedad Matemática Mexicana* **19** (1974), no. 1, 9–13.
43. D. Roberts, *Density of cubic field discriminants*, *Mathematics of Computation* **70** (2001), no. 236, 1699–1705.
44. B. E. Sagan, *The symmetric group. Representations, combinatorial algorithms, and symmetric functions*, Springer, 2000.
45. F.-O. Schreyer, *Syzygies of canonical curves and special linear series*, *Mathematische Annalen* **275** (1986), 105–137.

46. F.-O. Schreyer and F. Tanturri, *Matrix factorizations and curves in \mathbf{P}^4* , Documenta Mathematica **23** (2018), 1895–1924.
47. J.-P. Serre, *Linear representations of finite groups*, Springer New York, NY, 1977.
48. P. Solomatin, *Global fields and their L -functions*, Ph.D. thesis, Leiden University, 2021.
49. T. Taniguchi and F. Thorne, *Secondary terms in counting functions for cubic fields*, Duke Mathematical Journal **162** (2013), no. 13, 2451–2508.
50. Ravi Vakil, *Twelve points on the projective line, branched covers, and rational elliptic fibrations*, Mathematische Annalen **320** (1999), 33–54.
51. B. L. van der Waerden, *Die Zerlegungs- und Trägheitsgruppe als Permutationsgruppen*, Mathematische Annalen **111** (1935), 731–733.
52. B. van Geemen, *Some remarks on Brauer groups of $K3$ surfaces*, Advances in Mathematics **197** (2005), no. 1, 222–247.
53. S. Vemulapalli, *Bounds on successive minima of orders in number fields and scrollar invariants of curves*, arXiv preprint (2022), <https://arxiv.org/abs/2207.10522>.
54. F. Vermeulen, *Lifting curves of low gonality*, Master’s thesis, KU Leuven, 2019.
55. ———, *Arithmetically equivalent number fields have approximately the same successive minima*, arXiv preprint (2022), <https://arxiv.org/abs/2206.13855>, v2.
56. K. H. Wilson, *Three perspectives on n points in \mathbf{P}^{n-2}* , Ph.D. thesis, Princeton University, 2013.
57. M. M. Wood, *Moduli spaces for rings and ideals*, Ph.D. thesis, Princeton University, 2009.
58. A. Yukie, *Shintani zeta functions*, Cambridge University Press, 1993.
59. Y. Zhao, *On sieve methods for varieties over finite fields*, Ph.D. thesis, University of Wisconsin–Madison, 2013.

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