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Finite Representations of the Wright Function

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Abstract: The two-parameter Wright special function is an interesting mathematical object that arises in the theory of the space and time-fractional diffusion equations. Moreover, many other special functions are particular instantiations of the Wright function. The article demonstrates finite representations of the Wright function in terms of sums of generalized hypergeometric functions, which in turn provide connections with the theory of the Gaussian, Airy, Bessel, and Error functions, etc. The main application of the presented results is envisioned in computer algebra for testing numerical algorithms for the evaluation of the Wright function.

Keywords: Wright function; hypergeometric function; Bessel function; Error function; Airy function; Gaussian function

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1. Introduction

The Wright function was introduced in two seminal publications by the British mathematician Sir E.M. Wright, discussing the theory of partitions of numbers [1,2]. The function received renewed interest from the mathematical community when it was demonstrated that the space–time fractional diffusion equation with the temporal Caputo derivative can be solved in terms of Wright functions [3]. It was also discovered that the Wright function provides a unified treatment of several classes of special functions, notably the Bessel functions, the probability integral erf, the Airy Ai, Bi, and the Whittaker functions, among others. The Wright function was originally defined by the infinite series [1]:

$$W(a,b|\, z) := \sum_{k=0}^{+\infty} \frac{z^k}{k!\,\Gamma(ak+b)}, \quad z \in \mathbb{C},$$

under the conditions $b \in \mathbb{C}$ and $a > -1$, where Γ denotes Euler's Gamma function. Later works on the function include the articles of Gorenflo, Luchko, and Mainardi [4], and Luckko [5] among some others. Based on the sign of its first parameter, later, Mainardi classified the function into two types: the Wright function of the first type, if $a \geq 0$, and the Wright function of the second type, for $-1 < a < 0$ [6,7]. This function fits into the more general theory of the Fox–Wright (FW) functions as will be discussed in Section 3.

The Wright function is closely related to the theory of the generalized hypergeometric (GHG) functions. Notably, for rational parameter values, the Wright function can be represented as a finite sum of GHG functions. The link comes directly via the theory of Euler's Gamma function. Formulas for the Wright function representation of the first type have been published in [4,8] and have been derived via its representation as a Meijer G function. Recently, Apelblat and Gonzales-Santander have tabulated representations in terms of GHG functions for many parameter combinations [9].

The contribution of the present article is twofold. In the first place, it extends the results of the above authors [9] for the cases wherever $a < 0$ and $b > 1$ and also demonstrates how the domain of the first parameter can be extended into the negative integers under certain conditions by explicitly constructing polynomial representations of the function. These representations allow us to distinguish a Wright function of the third type (Section 4). Some of the present results have been presented in a preliminary form at the 2023 International Conference on Fractional Differentiation and Its Applications ICFDA, 2023 [10]. In the second place, the article exhibits the link with the Mittag-Leffler function, which also has wide applications in fractional calculus. It is demonstrated that the theory of the Wright function is very rich and can produce many potentially useful integral identities. In a similar way, the domain of the Mittag-Leffler function can be analytically continued into negative integral values of its first parameter and integer values of its second parameter.

2. Some Applications of the Wright Function

Recent surveys about Wright function applications can be found in [7,11]. What makes the function useful for applications in calculus is the fact that it is closed under differentiation

$$\frac{d}{dz}W(a,b|z) = W(a,a+b|z) \tag{1}$$

which allows one to write the integrals

$$W(a,b|z) = \int_0^z W(a,a+b|z) + \frac{1}{\Gamma(b)} \tag{2}$$

and

$$\int W(a,b|z)dz = W(a,b-a|z) + C \tag{3}$$

The Wright function arises in the theory of the space–time fractional diffusion equation (FDE) with the temporal Caputo derivative [3]. We recall that the Caputo's fractional derivative of order $\beta > 0$ is defined for $\beta \notin \mathbb{N}$ as the differintegral

$$\mathcal{D}_t^\beta f(t) := \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(u)du}{(t-u)^{\beta+1-m}} \tag{4}$$

where $m = \lfloor \beta \rfloor$. The fractional differential equation in the Caputo sense with variable coefficients

$$\mathcal{D}_t^b \left[t^b f'(t) \right] = a t^{b-1} f(t) \tag{5}$$

admits for a solution $f(t) = W(a,b|t^a)$ [12].

3. The Wright Function as a Simple Representative of the Fox–Wright Function Family

The generalized hypergeometric functions are defined by the infinite hypergeometric (HG) series

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q,x) := \sum_{m=0}^{+\infty} \frac{x^m}{\Gamma(m+1)} \prod_{k=1}^{p} \frac{\Gamma(a_k+m)}{\Gamma(a_k)} \prod_{k=1}^{q} \frac{\Gamma(b_k)}{\Gamma(b_k+m)} = \sum_{r=0}^{+\infty} \frac{x^r}{r!} \frac{\prod_{j=0}^{p-1} (a_j)_r}{\prod_{j=0}^{q-1} (b_j)_r} \tag{6}$$

where $(a)_r$ and $(b)_r$ will denote rising factorials and $(a)_0 = 1$, which assumes the normalization ${}_pF_q\left(\sim;\sim \middle| 0\right) = 1$. By convention, equal parameters in the numerator and denominator cancel out. Unless stated otherwise, it will be always assumed that the infinite series converge in some domain $x \in \mathbb{R}$.

The defining property for the HG series is that their coefficients are rational functions of the index variable (i.e., k). In the present article, we will use parametric notation similar to the one adopted by Oldham and Spanier [13].

$$
{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q,x) \equiv \left[\begin{array}{c|c} a_1,\ldots,a_p & x \\ b_1,\ldots,b_q & \end{array} \right]
$$

The FW functions are further generalizations of the GHG functions (in short hypergeometric functions) that can be defined by the infinite series

$$
{}_p\bar{\Psi}_q(x) \equiv \bar{\Psi}\left[\begin{array}{c|c} (A_1,a_1)\ldots,(A_p,a_p) & x \\ (B_1,b_1)\ldots,(B_q,b_q) & \end{array} \right] := \sum_{m=0}^{+\infty} \frac{x^m}{\Gamma(m+1)} \prod_{k=1}^{p} \frac{\Gamma(a_k m + A_k)}{\Gamma(A_k)} \prod_{k=1}^{q} \frac{\Gamma(B_k)}{\Gamma(b_k m + B_k)}
$$

whenever it converges.

At this point, the following extended tabular notation is introduced under the convention [14]

$$
{}_{p+1}\bar{\Psi}_q(z) \equiv \left[\begin{array}{c|c|c} a_1,\ldots,a_p & (A,a) & z \\ b_1,\ldots,b_q & - & \end{array} \right], \quad {}_{p+1}\bar{\Psi}_q(0) = 1,
$$

where the dash indicates absensce of Gamma factors in the series in the denominators and vice-versa. In this notation, the hypergeometric parameters of the function are written first while the composite parameters are left second. The right parameters result in factors of the form

$$
\frac{\Gamma(ka+A)}{\Gamma(A)}
$$

or their reciprocals, respectively, while the left parameters result in Pochhammer multipliers (i.e., $A \in \mathbb{N}$). The non-simplified parameters follow the usual convention established in the literature. The order in the parametric convention for the arguments of the Gamma function follows the usual convention.

The following simplifying convention will be used further:

$$
\left[\begin{array}{c|c|c} a_1,\ldots & - & z \\ b_1,\ldots & - & \end{array} \right] \equiv \left[\begin{array}{c|c} a_1,\ldots & z \\ b_1,\ldots & \end{array} \right] \tag{7}
$$

and

$$
\left[\begin{array}{c|c|c} a_1,\ldots,a_p & (A,1) & z \\ b_1,\ldots,b_q & - & \end{array} \right] \equiv \left[\begin{array}{c|c} a_1,\ldots,a_p,A & z \\ b_1,\ldots,b_q & \end{array} \right] \tag{8}
$$

This example shows different ways to write a hypergeometric function. Under this notation

$$
W(a,b|z) = \frac{1}{\Gamma(b)} \left[\begin{array}{c|c|c} - & - & z \\ - & (b,a) & \end{array} \right] \tag{9}
$$

In this way, one could appreciate that the Wright function is the simplest member of the class of the Fox–Wright functions. Other examples are the Bessel J function:

$$
J_\nu(z) = \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \left[\begin{array}{c|c} - & -\frac{z^2}{4} \\ \nu+1 & \end{array} \right]
$$

The Struve H function:

$$
H_\nu(z) = \frac{1}{\Gamma(\nu+3/2)\Gamma(3/2)} \left(\frac{z}{2}\right)^{\nu+1} \left[\begin{array}{c|c} 1 & -\frac{z^2}{4} \\ 3/2,\nu+3/2 & \end{array} \right]
$$

Furthermore, the following integral representation can be derived (see for example [14]):

$$\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array} \middle| \begin{array}{c} \ldots \\ (B,b) \end{array} \middle| z \right] = \frac{\Gamma(B)}{2\pi i}\int_{Ha^-} \frac{e^\tau}{\tau^B} \left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array} \middle| \begin{array}{c} \ldots \\ \ldots \end{array} \middle| \frac{z}{\tau^b} \right] d\tau \quad (10)$$

where Ha^- denotes the Hankel contour, which surrounds all poles of the GHG function from the left. Applied to the Wright function, where $B \mapsto a; b \mapsto a$, this gives the integral

$$W(a,b|z) = \frac{1}{2\pi i}\int_{Ha^-} \frac{e^{\xi + z\xi^{-a}}}{\xi^b} d\xi, \quad z \in \mathbb{C} \quad (11)$$

along a Hankel contour, which surrounds the negative real semi-axis and the pole at the origin. Said contour can be deformed in an extreme, as depicted in Figure 1. This contour consists of the rays AB and DE as well as the arc BCD. For integral values of b and a, the path of integration closes around the origin O so that the rays collapse and can be used to extend the domain of the function into the negative integer parameters.

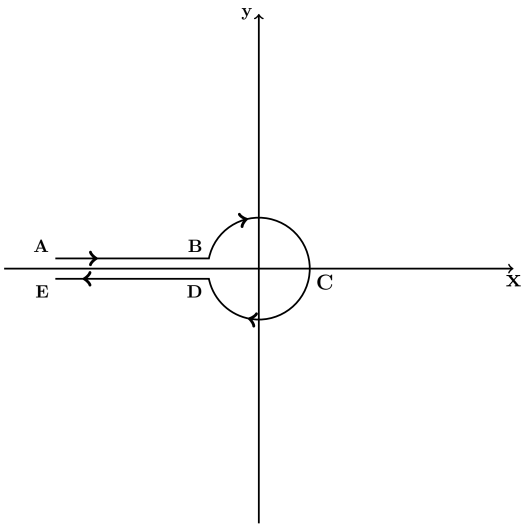

Figure 1. Partition of the Hankel contour.

4. Polynomial Reduction

In particular, let us consider the case when a is a negative integer and denote it by $-n$. Trivially, if b is a negative integer, say $b = -m$, then the above integral vanishes and $W(-n,-m|z) = 0$.

In contrast, if $n = -a$ and $b = m, m$, such that $n \in \mathbb{N}$, then

$$W(-n,m|z) = \mathrm{Res}(ker(\xi), \xi = 0) = \frac{1}{\Gamma(m)} \left\{ \left(\frac{d}{d\xi} \right)^{m-1} e^{\xi + z\xi^n} \right\} \bigg|_{\xi=0}$$

Therefore, we can conclude that $W(-n,m|z)$ is a polynomial in z. This is a novel result, which was not anticipated by Wright and allows for the extension of the domain of the parameters of the function. This polynomial can be computed explicitly by application of Faá di Bruno's formula using the complete exponential Bell polynomials. For the natural numbers n and m:

$$W(-n,m|z) = \frac{1}{\Gamma(m)} B_{m-1}\left(g'(0), \ldots, g^{(m-1)}(0) \right) \quad (12)$$

where $g(\xi) = \xi + z\xi^n$ is the exponent of the kernel and can be computed by the determinant

$$B_m\left(g'(0), \dots, g^{(m)}(0)\right) = \begin{vmatrix} \binom{m-1}{0}g'(0) & \binom{m-2}{1}g''(0) & \dots & \binom{m-1}{m-1}g^{(m)}(0) \\ -1 & \binom{m-2}{0}g'(0) & \dots & \binom{m-2}{m-2}g^{(m-1)}(0) \\ 0 & -1 & \dots & \binom{m-3}{m-3}g^{(m-3)}(0) \\ & & \dots & \\ 0 & \dots & -1 & \binom{0}{0}g'(0) \end{vmatrix}$$

Remark 1. *It should be noted that the resulting matrix is a band matrix since already $g''(0) = 0$. For example, for $n = 3$, $m = 8$, we have*

$$B_7(\dots) = \begin{vmatrix} 1 & 0 & 90z & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 60z & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 36z & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 18z & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 6\,z \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{vmatrix}$$

The polynomial reduction formulas allow us to claim that Mainardi's classification can be extended to add also Wright functions of the third type, that is whenever $a, b \in \mathbb{Z}^-$.

5. Finite Hypergeometric Representations

Wherever the a parameter is rational, the Wright function can be represented by a finite sum of hypergeometric functions. For positive and rational a, one could obtain the representation in terms of $\;_0F_{m+n-1}$ GHG functions [9]:

Theorem 1 (First HG Representation)**.** *Suppose that $a = n/m > 0$, where n and m are co-prime and $b \neq 0$. Then, $W(n/m, b|z)$ admits the finite representation*

$$W(n/m, b|z) = \sum_{r=0}^{m-1} \frac{z^r}{r!\,\Gamma(b + ar)} \left[\begin{array}{c|c} 1 & \frac{z^m}{n^n m^m} \\ \vec{b}, \vec{c} & \end{array} \right], \tag{13}$$

where $\vec{b}$ has n components and $\vec{c}$ has m components given by

$$b_j = r/m + (b + j)/n, \quad c_j = (r + 1 + j)/m,$$

respectively.

The proof follows [8] and is given as a staring point for the proof of the Second Representation Theorem.

Proof. Since the series is absolutely convergent we can arrange it in a finite number of ways. Starting from $a = n/m$, rearrange the series as

$$W(a, b|z) = \sum_{k=0}^{+\infty} \frac{z^k}{k!\,\Gamma(ak + b)} = \sum_{q=0}^{m-1} \sum_{p \geq q/m}^{+\infty} \frac{z^{mp-q}}{\Gamma(mp - q + 1)\,\Gamma(a(mp - q) + b)}$$

since the integer k can be partitioned as $k = mp - q$, where $q = 0, \dots m - 1$. After some algebra, we obtain

$$W(n/m, b|z) = \frac{1}{\Gamma(b)} + \sum_{r=1}^{m} z^r \sum_{p=0}^{+\infty} \frac{z^{mp}}{\Gamma(ap + ra + b)\Gamma(mp + r + 1)}.$$

Observe that for $p = 0$, the inner series evaluates to

$$C_r = \Gamma(ra + b)\Gamma(r + 1),$$

which serves as its normalization factor. Therefore, the series transforms as

$$W(n/m, b|z) = \sum_{r=0}^{m} \frac{z^r}{C_r} \sum_{p=0}^{+\infty} \frac{\Gamma(ra + b)\Gamma(r + 1)}{\Gamma(n(p + r/m + b/n))} \cdot \frac{z^{mp}}{\Gamma(m(p + (r + 1)/m))} \tag{14}$$

Further, use Proposition A1 in Appendix B to obtain

$$\frac{\Gamma(n(p + r/m) + b)}{\Gamma(nr/m + b)} = n^{np} \prod_{j=0}^{n-1} \underbrace{\left(\frac{r}{m} + \frac{b}{n} + \frac{j}{n}\right)_p}_{b_j} \tag{15}$$

From where we read off the base component

$$b_0 = \frac{r}{m} + \frac{b}{n}$$

with an increment $1/n$. Furthermore,

$$\frac{\Gamma(mp + r + 1)}{\Gamma(r + 1)} = m^{mp} \prod_{j=0}^{m-1} \underbrace{\left(\frac{r + 1}{m} + \frac{j}{m}\right)_p}_{c_j} \tag{16}$$

From where we read off the base component

$$c_0 = \frac{r + 1}{m}$$

with an increment $1/m$. □

Observe that $r = m - 1$ results in $c_1 = 1$. Therefore, the GHG functions reduce to the form ${}_0F_{m+n-1}$. The formula for a negative rational $a < 0$ needs some more work. Suppose first that $b < 1$. Let

$$W(-n/m, b|z) = \frac{1}{\Gamma(b)} + \sum_{r=1}^{m} \frac{z^r}{C_r} \sum_{p=0}^{+\infty} \frac{C_r}{\Gamma(-np - rn/m + b)} \frac{z^{mp}}{\Gamma(mp + r + 1)}$$

First, we use the Gamma reflection formula to obtain

$$\frac{1}{\Gamma(-np - rn/m + b)} = \frac{(-1)^{np}\Gamma\left(\frac{nr}{m} + np - b + 1\right)}{\Gamma\left(b - \frac{nr}{m}\right)\Gamma\left(\frac{nr}{m} - b + 1\right)} \tag{17}$$

Therefore,

$$W(-n/m, b|z) = \frac{1}{\Gamma(b)} + \sum_{r=1}^{m} \frac{z^r}{C_r} \sum_{p=0}^{+\infty} \frac{(-1)^{np} C_r \Gamma\left(\frac{nr}{m} + np - b + 1\right) z^{mp}}{\Gamma\left(b - \frac{nr}{m}\right)\Gamma\left(\frac{nr}{m} - b + 1\right)\Gamma(m(p + (r + 1)/m))} =$$

$$\frac{1}{\Gamma(b)} + \sum_{r=1}^{m} \frac{z^r}{C_r} \sum_{p=0}^{+\infty} \frac{(-1)^{np}\Gamma\left(\frac{nr}{m} + np - b + 1\right)\Gamma(r + 1) z^{mp}}{\Gamma\left(\frac{nr}{m} - b + 1\right)\Gamma(mp + r + 1))}$$

according to Equation (17). We use Proposition A1 to compute

$$\frac{\Gamma(n(r/m+p)-b+1)}{\Gamma\left(\frac{nr}{m}-b+1\right)} = n^{np}\prod_{j=0}^{n-1}\left(\frac{r}{m}+\frac{1-b}{n}+\frac{j}{n}\right)_p = n^{np}\prod_{j=n-1}^{0}\underbrace{\left(1+\frac{r}{m}-\frac{j+b}{n}\right)_p}_{b'_j}$$

Finally, we read off the parameters $b'_j = 1 + r/m - (b+j)/n$ with an increment $1/n$. Then, we can formulate the following

Theorem 2 (Second HG Representation)**.** *For $b \leq 1$ and $n \leq m$, non-negative co-prime integers, and $a = -n/m$,*

$$W(-n/m, b|z) = \sum_{r=0}^{m-1}\frac{z^r}{r!\,\Gamma(b+ar)}\left[\begin{array}{c|c} 1, \vec{b}' & \frac{(-)^n z^m}{n^n m^m} \\ \vec{c} & \end{array}\right] \tag{18}$$

where $\vec{b}' = \{b'_0 \dots b'_{n-1}\}$, $\vec{c} = \{c_0 \dots c_{m-1}\}$ and

$$b'_j = 1 + r/m - (b+j)/n, \quad c_j = (r+1+j)/m$$

Observe that for $r = m-1$ $c_1 = 1$; therefore, the GHG functions reduce to the form ${}_nF_{m-1}$. For $b \geq 1$, a polynomial part P must also be added to the representation as follows.

Theorem 3 (Third HG representation)**.** *Suppose that a and b are rational parameters, where $b \geq 1$ and $|a| \leq 1$. Define the polynomial $P_b(-a,z)$ by the integral recursion*

$$P_b(-a,z) := \int_0^z P_{b-a}(-a,x)dx + c_{b-1}, \tag{19}$$

where $c_{b-1} = 1/(b-1)!$ if b is an integer and 0 otherwise. Furthermore, define $P_0(z,-a) := 1$ and for $b < 0$ assign $P_b(z,-a) := 0$ identically. Then, for $a = -n/m$ and $b \neq 0$

$$W(-n/m, b|z) = \sum_{r=0}^{m-1}\frac{z^r}{r!\,\Gamma(b+ar)}\left[\begin{array}{c|c} 1, \vec{b}' & \frac{(-)^n z^m}{n^n m^m} \\ \vec{c} & \end{array}\right] + P_b(-a,z) \tag{20}$$

where m and n are co-prime numbers.

Proof. First, we prove that the arc integral results in a polynomial in z. Suppose that $b \geq 1$ is rational and $-a = n/m$ as before. Consider the arc BCD (Figure 1). We change variables as $\xi = \epsilon\eta^m$, $\epsilon > 0$. Then, the integral becomes

$$I = m\oint_{BCD}\frac{d\eta\,\epsilon^{1-b}}{\eta^{(b-1)m+1}}e^{\epsilon^{\frac{n}{m}}\eta^n z + \epsilon\eta^m}$$

The development of the kernel in the infinite series results in

$$ker = m\frac{d\eta\,\epsilon^{1-b}}{\eta^{(b-1)m+1}}\sum_{j=0}^{+\infty}\sum_{i=0}^{j}\frac{\epsilon^{\frac{im+(j-i)n}{m}}\eta^{im+(j-i)n}z^{j-i}}{i!(j-i)!}$$

The scale-invariant part of the series is given by the members k_j for which

$$\epsilon^{\frac{im+(j-i)n}{m}-b+1} = 1$$

This is given by the constraint

$$i = \frac{(b-1)m - jn}{m-n}$$

Therefore,

$$k_j = \frac{d\eta\ m\ z^{\frac{(j-b+1)n}{n-m}}}{\eta\left(\frac{(b-1)n-jm}{n-m}\right)!\left(\frac{(j-b+1)n}{n-m}\right)!}$$

Changing again the variables to $\eta = e^{i\varphi/m}$ results in the integral

$$c_j = \frac{1}{2\pi i}\frac{iz^{\frac{(j-b+1)m}{m-n}}}{\left(\frac{(b-1)m-jn}{m-n}\right)!\left(\frac{(j-b+1)m}{m-n}\right)!}\cdot\int_{-\pi}^{\pi} e^{\frac{i(bn-1)\varphi}{n}+\frac{i\varphi}{n}-ib\varphi}d\varphi = \frac{z^{\frac{(j-b+1)m}{m-n}}}{\left(\frac{(b-1)m-jn}{m-n}\right)!\left(\frac{(j-b+1)m}{m-n}\right)!}$$

Furthermore, the valid indices are given by the union set

$$j: \quad \left(\frac{(j-b+1)m}{m-n} \in \mathbb{N}\right) \cup \left(\frac{(b-1)m-jn}{m-n} \in \mathbb{N}\right)$$

Equivalently, in the a-notation

$$c_j = \frac{z^{\frac{j-b+1}{1-a}}}{\left(\frac{j-b+1}{1-a}\right)!\left(\frac{-aj+b-1}{1-a}\right)!} \tag{21}$$

Therefore, $a < 1$ must hold for c_j not to vanish.

On the other hand,

$$b-1 \le j \le (b-1)/a \quad \cup \quad j \in \mathbb{N}$$

which is a finite set. Therefore, for a rational b, the integral I is a polynomial in z.

To derive the polynomial recursion, we proceed from Equation (2)

$$W(-a,b|z) = \int_0^z W(-a,b-a|z) + \frac{1}{\Gamma(b)} \tag{22}$$

so that the equation defines a recursion relationship.

Observe that for $j = b-1 \in \mathbb{N}$, the coefficient becomes

$$c_{b-1} = \frac{1}{(b-1)!} = \frac{1}{\Gamma(b)}$$

Therefore, for non-integer b, there are no constant monomials. Furthermore, consider the monomial c_j as a function of b. Differentiating Equation (21), one obtains the recursion

$$\frac{d}{dz}c_j(b) = \frac{z^{\frac{j-b+1}{1-a}-1}}{\left(\frac{j-b+1}{1-a}-1\right)!\left(\frac{-aj+b-1}{1-a}\right)!} = \frac{z^{\frac{j-(b-a)}{1-a}}}{\left(\frac{j-(b-a)}{1-a}\right)!\left(\frac{aj-b+1}{a-1}\right)!} = c_{j-1}(b-a),$$

which is also consistent with the integral Equation (22). Therefore, the polynomial $P_b(-a,z)$ should obey the above recursion. The second argument of the Wright function varies; therefore, it is convenient that it indexes the polynomial in a slight notational change. □

For integer values of a, that is, when $m = 1$, Theorem 3 corresponds with the polynomial representation since the hypergeometric sum disappears.

6. The Special Case $b = 0$

The case whenever $b = 0$ needs separate treatment. From the theory of the FW functions, we can formulate the following proposition.

Proposition 1. *Whenever $b = 0$, we have the special FW representation*

$$W(a, 0|z) = \frac{z}{\Gamma(a)} \left[\begin{array}{c|c|c} 1 & - & z \\ 2 & (a, a) & \end{array} \right] \tag{23}$$

Proof. The proof follows via direct computation:

$$W(a, 0|z) = \sum_{k=1}^{+\infty} \frac{z^k}{\Gamma(ak)\Gamma(k+1)} = \sum_{j=0}^{+\infty} \frac{z^{j+1}}{\Gamma(aj+a)\Gamma(j+2)} = \frac{z}{\Gamma(a)} \left[\begin{array}{c|c|c} 1 & - & z \\ 2 & (a, a) & \end{array} \right]$$

□

This result can be represented for rational a using the theory developed so-far as follows.

Proposition 2. *Whenever $a = m/n > 0$ with n, m co-prime natural numbers*

$$W(a, 0|z) = \sum_{r=0}^{m-1} \frac{z^{r+1}}{(r+1)!\,\Gamma(a+ar)} \left[\begin{array}{c|c} 1 & \frac{z^m}{n^n m^m} \\ \vec{b}, \vec{c} & \end{array} \right] \tag{24}$$

where $\vec{b}$ has n components and $\vec{c}$ has m components given by

$$b_j = (r+1)/m + j/n, \quad c_j = (r+2+j)/m,$$

respectively.

Proof. Starting from Equation (14), we observe that $C_r = \Gamma(ar+a)\Gamma(r+2)$. Furthermore,

$$\frac{\Gamma(mp+r+2)}{\Gamma(r+2)} = m^{mp} \prod_{j=0}^{m-1} \underbrace{\left(\frac{r+2}{m} + \frac{j}{m} \right)_p}_{c_j} \tag{25}$$

by Proposition A1. From where we read off the component

$$c_0 = \frac{r+2}{m}$$

with an increment $1/m$. □

In a similar way, we can state

Proposition 3. *Whenever $-1 < a = n/m < 0$ with n, m co-prime natural numbers*

$$W(-n/m, 0|z) = \sum_{r=0}^{m-1} \frac{z^{r+1}}{(r+1)!\,\Gamma(a+ar)} \left[\begin{array}{c|c} 1, \vec{b}' & \frac{(-)^n z^m}{n^n m^m} \\ \vec{c} & \end{array} \right] \tag{26}$$

where $\vec{b}' = \{b'_0 \dots b'_{n-1}\}$, $\vec{c} = \{c_0 \dots c_{m-1}\}$ and

$$b'_j = 1 + (r+1)/m - j/n, \quad c_j = (r+2+j)/m$$

Proof. Use the Gamma reflection formula to obtain

$$b'_j = 1 - (-r/m + (-n/m + j)/n) = 1 + r/m - (-n/m + j)/n = 1 + (r+1)/m - j/n$$

□

7. Representations of the Wright Function of the First Type

The following representations can be computed using Theorem 2:

7.1. Representations for $a = 1/2$

The following representation holds.

$$W(1/2, b|z) = \frac{1}{\Gamma(b)} \left[\begin{array}{c|c} - & \frac{z^2}{4} \\ b, 1/2 & \end{array} \right] + \frac{z}{\Gamma(b+1/2)} \left[\begin{array}{c|c} - & \frac{z^2}{4} \\ b+1/2, 3/2 & \end{array} \right] \tag{27}$$

7.2. Representations for $a = 1/3$ and $a = 2/3$

Using the conventional notation we have

$$W(1/3, b|z) = \frac{{}_0F_3\left(-; b+\frac{2}{3}, \frac{4}{3}, \frac{5}{3}; \frac{z^3}{27}\right) z^2}{2\Gamma\left(b+\frac{2}{3}\right)} + \frac{{}_0F_3\left(-, b+\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; \frac{z^3}{27}\right) z}{\Gamma\left(b+\frac{1}{3}\right)} + \frac{{}_0F_3\left(-; b, \frac{1}{3}, \frac{2}{3}; \frac{z^3}{27}\right)}{\Gamma(b)} \tag{28}$$

While in the tabular notation:

$$W(2/3, b|z) = \frac{1}{\Gamma(b)} \left[\begin{array}{c|c} - & \frac{z^3}{108} \\ \frac{b}{2}, \frac{b}{2}+\frac{1}{2}, \frac{1}{3}, \frac{2}{3} & \end{array} \right] + \frac{z}{\Gamma\left(b+\frac{2}{3}\right)} \left[\begin{array}{c|c} - & \frac{z^3}{108} \\ \frac{b}{2}+\frac{1}{3}, \frac{b}{2}+\frac{5}{6}, \frac{2}{3}, \frac{4}{3} & \end{array} \right] + \frac{z^2}{2\Gamma\left(b+\frac{4}{3}\right)} \left[\begin{array}{c|c} - & \frac{z^3}{108} \\ \frac{b}{2}+\frac{2}{3}, \frac{b}{2}+\frac{7}{6}, \frac{4}{3}, \frac{5}{3} & \end{array} \right] \tag{29}$$

7.3. Relationship to Trigonometric and Bessel Functions

In a similar way as for the Bessel functions, for half-integer values of the second parameter, the Wright function can be represented by trigonometric functions as follows:

$$W\left(1, 1/2 | \frac{x^2}{4}\right) = \frac{\cosh(x)}{\sqrt{\pi}}, \quad W\left(1, 1/2 | -\frac{x^2}{4}\right) = \frac{\cos(x)}{\sqrt{\pi}}$$

and

$$W\left(1, 3/2 | \frac{x^2}{4}\right) = 2\frac{\sinh(x)}{\sqrt{\pi x}}, \quad W\left(1, 3/2 | -\frac{x^2}{4}\right) = 2\frac{\sin(x)}{\sqrt{\pi x}}$$

For $b = 0$, according to Equation (23), we have the special cases

$$W(1, 0|x) = I_1(2\sqrt{x})\sqrt{x}$$

and

$$W(1, 0| - x) = J_1(2\sqrt{x})\sqrt{x}$$

8. Representations of the Wright Function of the Second Type

The main application of Theorem 3 is the representation of Mainardi's function [15]

$$M_a(z) = W(-a, 1-a| - z)$$

The integral of the function is

$$IM_a(z) = -W(-a, 1| - z)$$

and its nth derivative is

$$M_a^{(n)}(z) = (-)^n W(-a, 1 - (n+1)a| - z)$$

8.1. Representations for a = −1/4

$$M_{1/4}(z) = W(-1/4, 3/4| - z) = \frac{{}_0F_2\left(-; \frac{5}{4}, \frac{3}{2}; -\frac{z^4}{256}\right) z^2}{2\Gamma\left(\frac{1}{4}\right)} - \frac{{}_0F_2\left(-; \frac{3}{4}, \frac{5}{4}; -\frac{z^4}{256}\right) z}{\sqrt{\pi}} + \frac{{}_0F_2\left(-; \frac{1}{2}, \frac{3}{4}; -\frac{z^4}{256}\right)}{\Gamma\left(\frac{3}{4}\right)} \tag{30}$$

8.2. Representations for a = −1/3

The general formula for $b \leq 1$ reads

$$W(-1/3, b|z) = \frac{{}_1F_2\left(\frac{5}{3} - b; \frac{4}{3}, \frac{5}{3}; -\frac{z^3}{27}\right) z^2}{2\Gamma\left(b - \frac{2}{3}\right)} + \frac{{}_1F_2\left(\frac{4}{3} - b; \frac{2}{3}, \frac{4}{3}; -\frac{z^3}{27}\right) z}{\Gamma\left(b - \frac{1}{3}\right)} + \frac{{}_1F_2\left(1 - b; \frac{1}{3}, \frac{2}{3}; -\frac{z^3}{27}\right)}{\Gamma(b)} \tag{31}$$

For $b = 1/3$ and $z > 0$, the equation reduces to

$$W(-1/3, 1/3| - z) = -\frac{z}{3}\left(I_{2/3}\left(\frac{2z^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right) - I_{-2/3}\left(\frac{2z^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right)\right) = \frac{z}{\sqrt{3}\pi} K_{2/3}\left(\frac{2z^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right) = -\sqrt[3]{3}\,\mathrm{Ai}'\left(\frac{z}{\sqrt[3]{3}}\right) \tag{32}$$

while

$$W(-1/3, 1/3|z) = -\frac{z}{3}\left(J_{2/3}\left(\frac{2z^{3/2}}{3^{3/2}}\right) - J_{-2/3}\left(\frac{2z^{3/2}}{3^{3/2}}\right)\right) = -\sqrt[3]{3}\,\mathrm{Ai}'\left(-\frac{z}{\sqrt[3]{3}}\right)$$

A plot is presented in Figure 2A. Regarding the Mainardi function $M_{1/3} = W(-1/3, 2/3| - z)$, Equation (31) simplifies as expected for $b = 2/3$ to the Airy Ai function, which can be represented as a weighted sum of Bessel J or I functions, respectively. That is, for $z > 0$

$$W(-1/3, 2/3| - z) = \frac{I_{-1/3}\left(\frac{2z^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right)\sqrt{z}}{\sqrt{3}} - \frac{I_{1/3}\left(\frac{2z^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right)\sqrt{z}}{\sqrt{3}} = \frac{K_{1/3}\left(\frac{2z^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right)\sqrt{z}}{\pi} = \sqrt[3]{3^2}\,\mathrm{Ai}\left(-z/\sqrt[3]{3}\right) \tag{33}$$

while for $z < 0$

$$W(-1/3, 2/3|z) = \frac{J_{-1/3}\left(\frac{2z^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right)\sqrt{z}}{\sqrt{3}} + \frac{J_{1/3}\left(\frac{2z^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right)\sqrt{z}}{\sqrt{3}} = \sqrt[3]{3^2}\,\mathrm{Ai}\left(-z/\sqrt[3]{3}\right) \tag{34}$$

A plot is presented in Figure 2A together with its antiderivatives—Figure 2B.

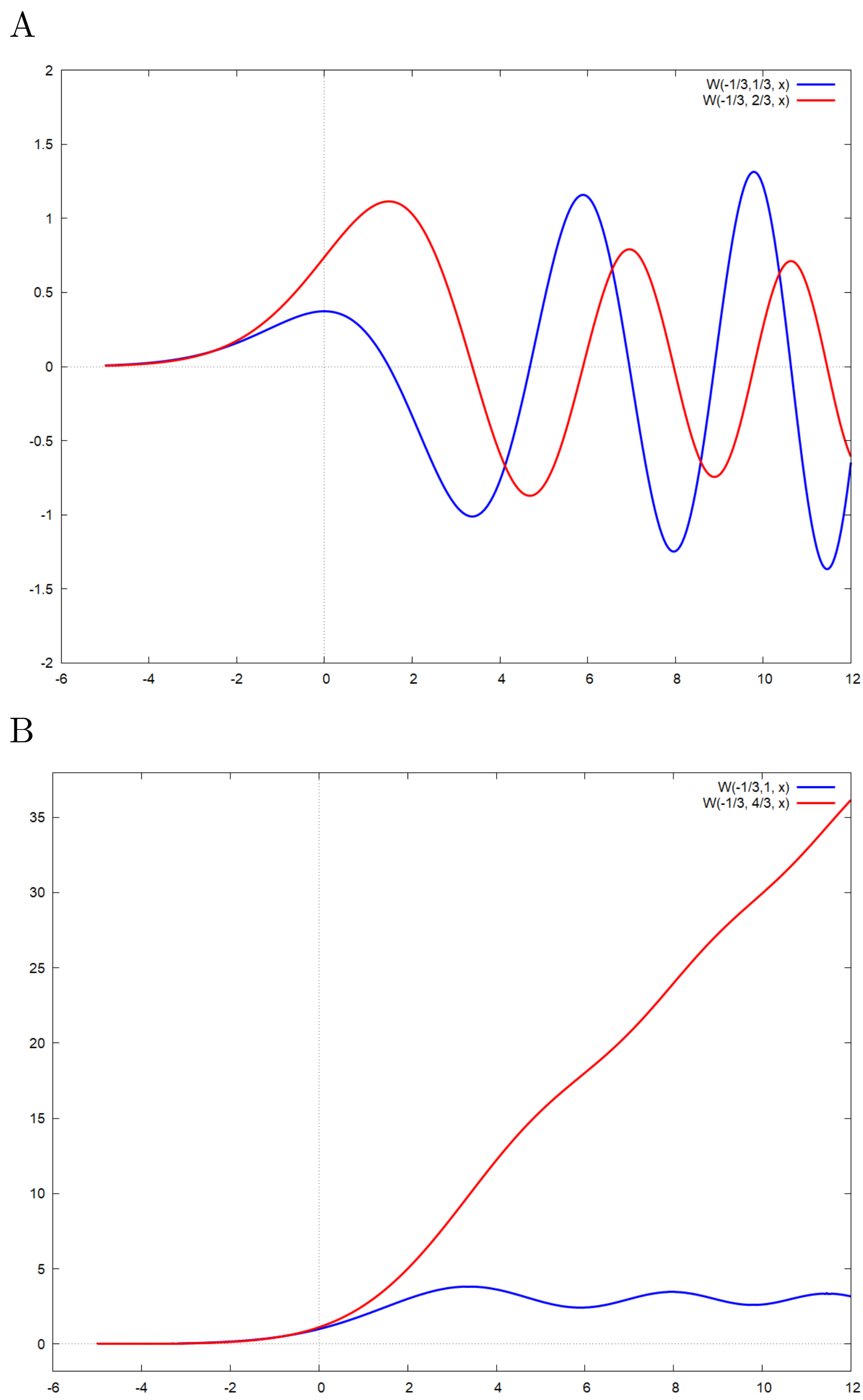

Figure 2. Plots of $W(a, b|x)$ for $a = -1/3$.

8.3. Representations for $a = -1/2$

For $b \leq 1$, we have the general representation in terms of Kummer M functions

$$W(-1/2, b|z) = \frac{{}_1F_1\left(\frac{3}{2} - b; \frac{3}{2}; -\frac{z^2}{4}\right)z}{\Gamma\left(b - \frac{1}{2}\right)} + \frac{{}_1F_1\left(1 - b; \frac{1}{2}; -\frac{z^2}{4}\right)}{\Gamma(b)} \tag{35}$$

For $b = 3/4$, $z > 0$, this relates to the Bessel K function:

$$W(-1/2, 3/4|-z) = \frac{\sqrt{z}}{\sqrt{2}\pi} K_{1/4}\left(\frac{z^2}{8}\right) e^{-\frac{z^2}{8}} \tag{36}$$

In particular, for $b = 1$, the above Equation (35) reduces to the complementary error integral

$$W(-1/2, 1|z) = 1 + \mathrm{erf}\left(\frac{z}{2}\right) = \mathrm{erfc}\left(-\frac{z}{2}\right)$$

in accordance with the polynomial reduction. A plot is presented in Figure 3A together with its derivative-the Gaussian kernel.

The Gaussian derivatives can be represented as

$$\left(\frac{d}{dz}\right)^n \frac{e^{-z^2/4}}{\sqrt{\pi}} = W\left(-\frac{1}{2}, \frac{1-n}{2}\middle| z\right) \tag{37}$$

Their plots are presented in Figure 3B togeter with the Gaussian kernel (Figure 3A). The anti-derivatives of the Gaussian kernel can be computed in a similar way using Theorem 3. For example, for $b = 7/2$

$$W(-1/2, 7/2|z) = \frac{1}{\sqrt{\pi}}\left(\frac{z^4}{60} + \frac{3z^2}{10} + \frac{8}{15}\right)e^{-\frac{z^2}{4}} + \left(1 + \mathrm{erf}\left(\frac{z}{2}\right)\right)\left(\frac{z^5}{120} + \frac{z^3}{6} + \frac{z}{2}\right) \tag{38}$$

A

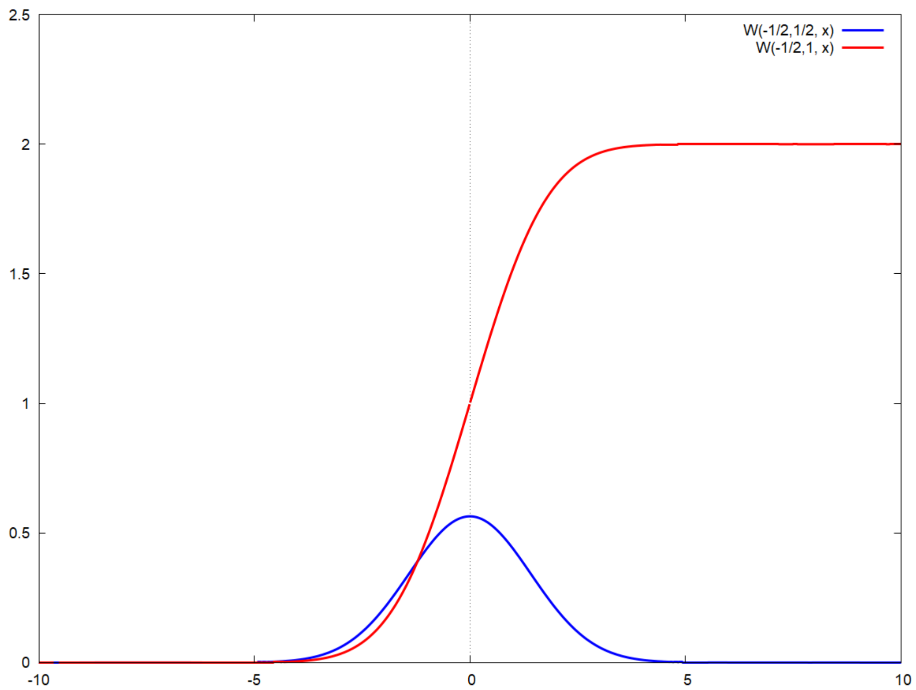

Figure 3. *Cont.*

B

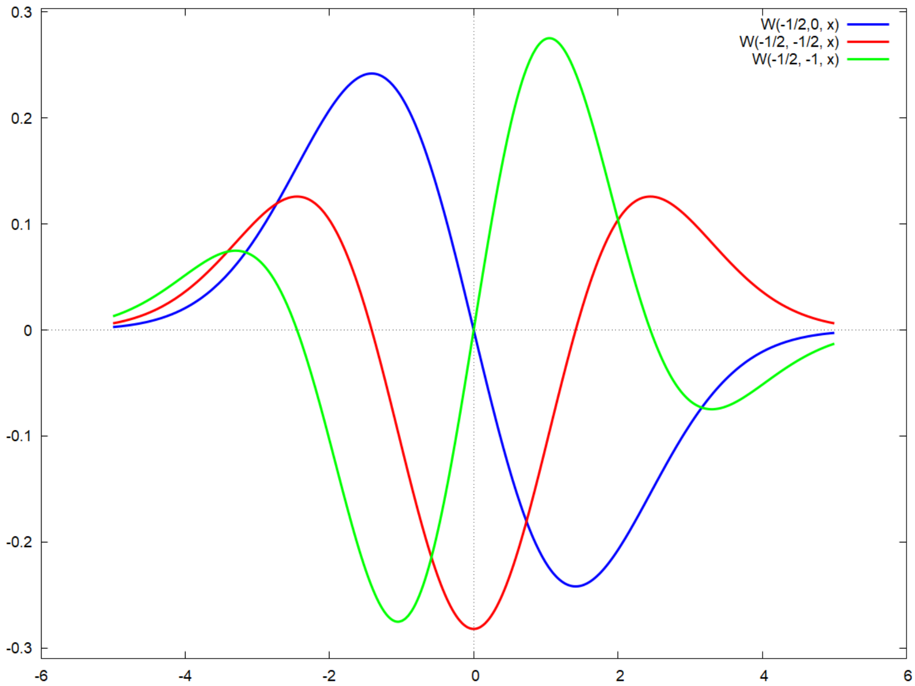

Figure 3. Plots of $W(a,b|x)$ for $a=-1/2$.

8.4. Representations for $a = -2/3$

The Mainardi function for $a=2/3$ can be represented as the difference of two exponentially weighted Bessel K functions on the entire real line as follows

$$W(-2/3,1/3|-z) = \frac{K_{2/3}\left(-\frac{2z^3}{27}\right)z^2e^{-\frac{2z^3}{27}}}{3^{\frac{3}{2}}\pi} - \frac{K_{1/3}\left(-\frac{2z^3}{27}\right)z^2e^{-\frac{2z^3}{27}}}{3^{\frac{3}{2}}\pi} \tag{39}$$

On the other hand,

$$W(-2/3,1/3|z) = -\frac{e^{\frac{2z^3}{27}}\left(3\,\mathrm{Ai}'\left(\frac{z^2}{3^{\frac{4}{3}}}\right)+\sqrt[3]{3}z\,\mathrm{Ai}\left(\frac{z^2}{3^{\frac{4}{3}}}\right)\right)}{3^{\frac{2}{3}}}$$

in terms of the Airy Ai function and its derivative.

For $b=2/3$

$$W(-2/3,2/3|z) = \frac{K_{1/3}\left(\frac{2z^3}{27}\right)ze^{\frac{2z^3}{27}}}{\sqrt{3}\pi} = \sqrt[3]{9}\,e^{\frac{2z^3}{27}}\,\mathrm{Ai}\left(\frac{z^2}{3^{\frac{4}{3}}}\right) \tag{40}$$

Plots are presented in Figure 4A toghter with their antiderivatives—Figure 4B.

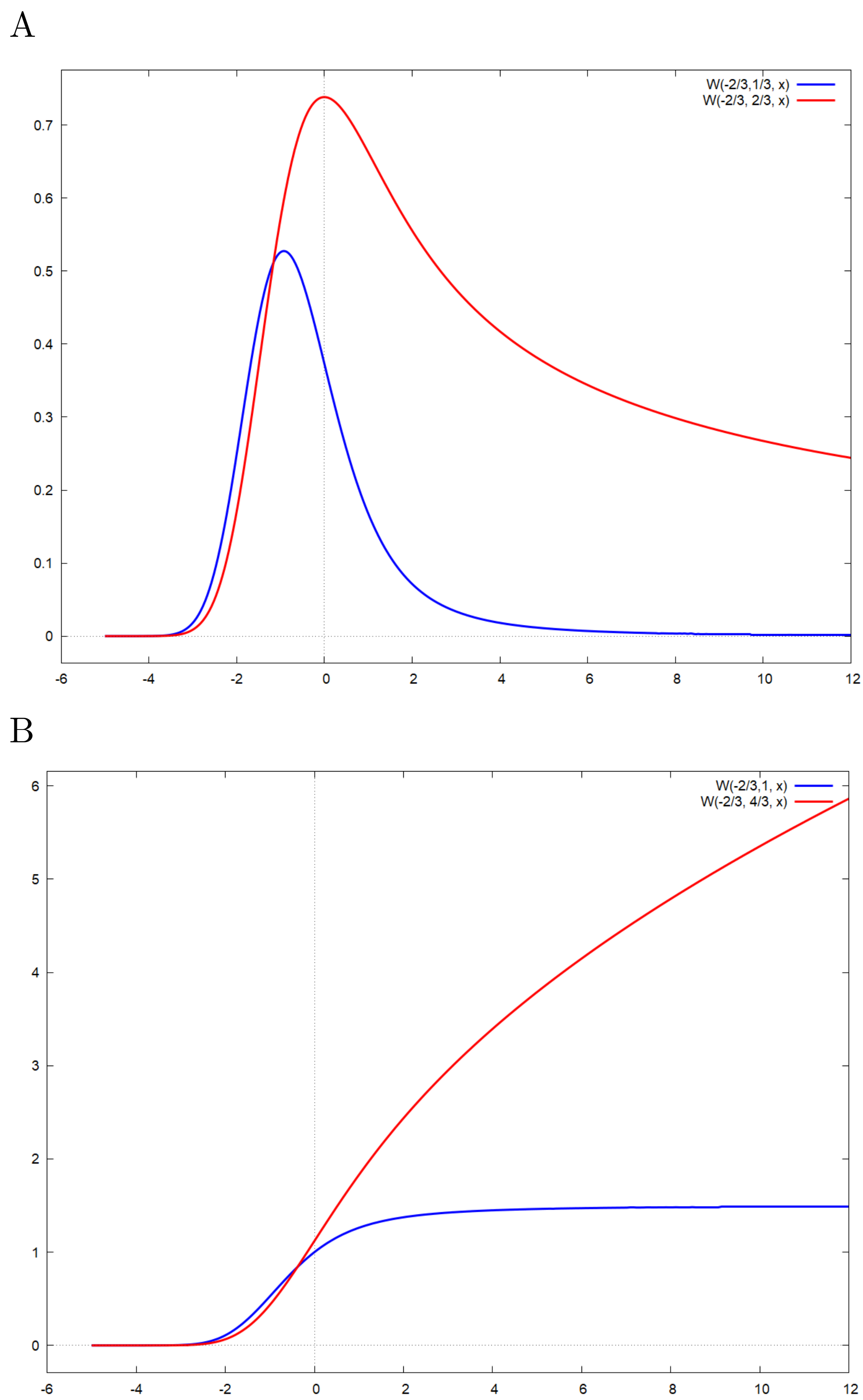

Figure 4. Plots of $W(a, b|x)$ for $a = -2/3$.

For $b = 4/3$

$$W(-2/3, 4/3|z) = -\frac{\left(I_{2/3}\left(\frac{2z^3}{27}\right) + I_{1/3}\left(\frac{2z^3}{27}\right) - I_{-1/3}\left(\frac{2z^3}{27}\right) - I_{-2/3}\left(\frac{2z^3}{27}\right)\right) z^2 e^{\frac{2z^3}{27}}}{3}$$

This can be further represented as

$$W(-2/3,4/3|z) = 9^{\frac{1}{3}} z e^{\frac{2z^3}{27}} \operatorname{Ai}\left(\frac{z^2}{3^{\frac{4}{3}}}\right) - 9^{\frac{2}{3}} e^{\frac{2z^3}{27}} \operatorname{Ai}'\left(\frac{z^2}{3^{\frac{4}{3}}}\right) \tag{41}$$

9. Representations of the Wright Function of the Third Type

9.1. Representations for $a = -1$

This formula was recently derived in [9] and is not anticipated in the previous literature since the parameter domain is customarily restricted to $a > -1$.

$$W(-1,b|z) = \frac{{}_1F_0(1-b;-;z)}{\Gamma(b)} = \frac{(z+1)^{b-1}}{\Gamma(b)} \tag{42}$$

9.2. Representations for $a \in \mathbb{Z}^-$

For negative integers, representations can be tabulated for some cases as follows. For $n = 1$:

$$1, z+1$$

$n = 2$:

$$\frac{1}{2}, \frac{2z+1}{2}, \frac{(z+1)^2}{2}$$

$n = 3$:

$$\frac{1}{6}, \frac{6z+1}{6}, \frac{6z+1}{6}, \frac{(z+1)^3}{6}$$

$n = 4$

$$\frac{1}{24}, \frac{24z+1}{24}, \frac{24z+1}{24}, \frac{12z^2+12z+1}{24}, \frac{(z+1)^4}{24}$$

$n = 5$:

$$\frac{1}{120}, \frac{120z+1}{120}, \frac{120z+1}{120}, \frac{60z+1}{120}, \frac{60z^2+20z+1}{120}, \frac{(z+1)^5}{120}$$

$n = 6$:

$$\frac{1}{720}, \frac{720z+1}{720}, \frac{720z+1}{720}, \frac{360z+1}{720}, \frac{360z^2+120z+1}{720}, \frac{120z^3+180z^2+30z+1}{720}, \frac{(z+1)^6}{720}$$

10. The Mittag-Leffler Function as a Laplace Transform of the Wright Function

The main application of the presented results so far is related to the Mittag-Leffler function $E_{a,b}(z)$. The two-parameter Mittag-Leffler function [16,17] under the present convention will be denoted as

$$E_{a,b}(z) := \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(ak+b)} = \frac{1}{\Gamma(b)} \left[\begin{array}{c|c|c} 1 & - & z \\ - & (b,a) & \end{array} \right], \quad a > 0,\ b \neq 0$$

This immediately gives the complex integral representation according to Equation (10)

$$E_{a,b}(z) = \frac{1}{2\pi i} \int_{Ha^-} \frac{e^\tau}{\tau^b} \left[\begin{array}{c|c} 1 & \frac{z}{\tau^a} \\ - & \end{array} \right] d\tau = \frac{1}{2\pi i} \int_{Ha^-} \frac{e^\tau}{\tau^b} \frac{d\tau}{1-\frac{z}{\tau^a}} = \frac{1}{2\pi i} \int_{Ha^-} \frac{\tau^{a-b} e^\tau}{\tau^a - z} d\tau \tag{43}$$

For real indices a_i and b_i, $A > 0$ and $a > 0$, it was proven that [14]

$$\left[\begin{array}{c|c|c} a_1,\ldots,a_p & (A,a) & z \\ b_1,\ldots,b_q & \ldots & \end{array} \right] = \frac{1}{\Gamma(A)} \int_0^{+\infty} e^{-\tau} \tau^{A-1} \left[\begin{array}{c|c|c} a_1,\ldots,a_p & \ldots & z\,\tau^a \\ b_1,\ldots,b_q & \ldots & \end{array} \right] d\tau \tag{44}$$

whenever the GHG function in the integral kernel converges. Then, by Equation (44) for $A = 1, a = 1$, it follows immediately that

$$E_{a,b}(z) = \int_0^{+\infty} e^{-t} W(a, b|zt) dt \tag{45}$$

This representation can be used to also derive a Laplace transform pair:

$$\frac{1}{s} E_{a,b}\left(\frac{1}{s}\right) = \frac{1}{s} \int_0^{+\infty} e^{-\tau} W\left(a, b|\frac{\tau}{s}\right) d\tau = \\ \frac{1}{s} \int_0^{+\infty} e^{-st} W\left(a, b|\frac{st}{s}\right) d(st) = \int_0^{+\infty} e^{-st} W(a, b|t) dt$$

for $s > 0$, since the integration variable $\tau = st$ is positive. Therefore,

$$W(a, b|t) \div \frac{1}{s} E_{a,b}\left(\frac{1}{s}\right), \quad a > 0 \tag{46}$$

On the other hand, for the Wright function of the second type, we have

$$\int_0^{+\infty} e^{-st} W(-a, b|t) dt = \frac{1}{2\pi i} \int_0^{+\infty} e^{-st} dt \int_{Ha^-} \frac{e^{\xi + t\xi^a}}{\xi^b} d\xi = \\ \frac{1}{2\pi i} \int_{Ha^-} \frac{e^{\xi}}{\xi^b} d\xi \int_0^{+\infty} e^{t(\xi^a - s)} dt = \frac{1}{2\pi i} \int_{Ha^-} \frac{e^{\xi}}{\xi^b(\xi^a - s)} d\xi = \\ \frac{1}{2\pi i} \int_{Ha^-} \frac{e^{\xi} \xi^{a-(a+b)}}{\xi^a - s} d\xi = E_{a,a+b}(s)$$

under the condition $\mathrm{Re}(\xi^a - s) < 0$. Therefore, the corresponding Laplace transform pair is

$$W(-a, b|t) \div E_{a,a+b}(s), \quad 0 \le a \le 1 \tag{47}$$

This gives the relationship between the Wright and Mittag-Leffler functions.

For every rational parameter pair, the ML function is reducible to a finite sum of HG functions as the following theorem [9]:

Theorem 4 (Mittag-Leffler HG Representation)**.** *Suppose that $a = n/m > 0$, where n and m are co-prime, and $b \neq 0$. Then*

$$E_{a,b}(z) = \sum_{r=0}^{m-1} \frac{z^r}{\Gamma(b + ar)} \left[\begin{array}{c|c} 1 & \frac{z^m}{n^n} \\ \vec{b} & \end{array} \right], \tag{48}$$

where $\vec{b}$ has n components

$$b_j = r/m + (b + j)/n$$

Proof. Starting from

$$E_{n/m,b}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(ak + b)} = \sum_{q=0}^{m-1} \sum_{p \ge q/m}^{+\infty} \frac{z^{mp-q}}{\Gamma(a(mp - q) + b)}$$

since the integer n can be partitioned as $k = mp - q$, where $q = 0, \ldots m - 1$. After some algebra, we obtain

$$E_{n/m,b}(z) = \frac{1}{\Gamma(b)} + \sum_{r=1}^{m} z^r \sum_{p=0}^{+\infty} \frac{z^{mp}}{\Gamma(ap + ra + b)}.$$

Observe that for $p = 0$, the inner series coefficient is $C_r = \Gamma(ra + b) = \Gamma(nr/m + b)$, which serves as a normalization factor. Therefore, the series transforms as

$$E_{n/m,b}(z) = \sum_{r=0}^{m} \frac{z^r}{C_r} \sum_{p=0}^{+\infty} \frac{C_r}{\Gamma(n(p + r/m) + b)} z^{mp} \tag{49}$$

Further, use Proposition A1 to obtain

$$\frac{\Gamma(n(p + r/m) + b)}{\Gamma(nr/m + b)} = n^{np} \prod_{j=0}^{n-1} \underbrace{(r/m + b/n + j/n)_p}_{b_j} \tag{50}$$

Therefore,

$$E_{n/m,b}(z) = \sum_{r=0}^{m} \frac{z^r}{\Gamma(ra + b)} \sum_{p=0}^{+\infty} \frac{z^{mp}}{n^{np} \prod_{j=0}^{n-1} b_j}$$

From where we read off

$$b_0 = \frac{r}{m} + \frac{b}{n}$$

with an increment $1/n$; so that

$$E_{n/m,b}(z) = \sum_{r=0}^{m-1} \frac{z^r}{\Gamma(b + ar)} \left[\begin{array}{c|c} 1 & \frac{z^m}{n^n} \\ \vec{b} & \end{array} \right]$$

□

Observe that for $r = m - 1$ $c_1 = 1$; therefore, the GHG functions reduce to the form ${}_0F_{m-1}$. Unlike for the Wright function, whenever $b = 0$

$$E_{a,0}(z) = \sum_{k=0}^{+\infty} \frac{z^{k+1}}{\Gamma(ak + a)} = \frac{z}{\Gamma(a)} \left[\begin{array}{c|c|c} 1 & - & z \\ - & (a, a) & \end{array} \right] = z E_{a,a}(z)$$

which is another Mittag-Leffler function. Therefore, the previous case directly applies.

$$E_{n/m,0}(z) = z \sum_{r=0}^{m-1} \frac{z^r}{\Gamma(a + ar)} \left[\begin{array}{c|c} 1 & \frac{z^m}{n^n} \\ \vec{b} & \end{array} \right], \quad a = n/m \tag{51}$$

where $\vec{b}$ has n components $b_j = (r + 1)/m + j/n$.

10.1. Some Integral Identities Interlinking the ML and Wright Functions

This allows one to write the following sets of integrals via the application of Equation (45): For $m, n > 0$, according to the First Representation Theorem

$$E_{n/m,b}(z) = \sum_{r=0}^{m-1} \frac{z^r}{\Gamma(b + ar)} \left[\begin{array}{c|c} 1 & \frac{z^m}{n^n} \\ \vec{b} & \end{array} \right] = \int_0^{+\infty} e^{-t} \sum_{r=0}^{m-1} \frac{z^r t^r}{r!\, \Gamma(b + ar)} \left[\begin{array}{c|c} 1 & \frac{z^m t^m}{n^n m^m} \\ \vec{b}, \vec{c} & \end{array} \right] dt = \sum_{r=0}^{m-1} \frac{z^r}{r!\, \Gamma(b + ar)} \int_0^{+\infty} t^r e^{-t} \left[\begin{array}{c|c} 1 & \frac{z^m t^m}{n^n m^m} \\ \vec{b}, \vec{c} & \end{array} \right] dt$$

for $b \neq 0$. Therefore, after the substitution $y = z^m/n^n$, we have

$$\left[\begin{array}{c|c} 1 & y \\ \vec{b} & \end{array} \right] = \frac{1}{\Gamma(r + 1)} \int_0^{+\infty} t^r e^{-t} \left[\begin{array}{c|c} 1 & y \frac{t^m}{m^m} \\ \vec{b}, \vec{c} & \end{array} \right] dt, \tag{52}$$

where $\vec{b} = \{r/m + (b+j)/n\}$, $\vec{c} = \{(r+1+j)/m\}$ as discussed above. The last formula can be used to produce many integral identities between GHG functions.

10.2. Analytical Continuation of the ML Function for Negative Parameters

The integral representation allows one to analytically continue the ML for negative first parameters. Then, one has formally

$$E_{-a,b}(z) := \frac{1}{2\pi i} \int_{Ha^-} \frac{e^{\tau}}{\tau^b} \frac{d\tau}{1 - z\tau^a}, \quad a > 0$$

Therefore, for rational parameters, we can apply the Second and Third Representation theorems to obtain for $|a| < 1$

$$E_{-n/m,b}(z) = \sum_{r=0}^{m-1} \frac{z^r}{r!\,\Gamma(b+ar)} \int_0^{+\infty} e^{-t} t^r \left[\begin{array}{c} 1, \vec{b} \\ \vec{c} \end{array} \middle| \, \frac{(-)^n z^m t^m}{n^n m^m} \right] dt \tag{53}$$

for $|b| < 1$ and

$$E_{-n/m,b}(z) = \sum_{r=0}^{m-1} \frac{z^r}{r!\,\Gamma(b+ar)} \int\limits_0^{+\infty} e^{-t} t^r \left[\begin{array}{c} 1, \vec{b} \\ \vec{c} \end{array} \middle| \, \frac{(-)^n z^m t^m}{n^n m^m} \right] dt + \int\limits_0^{+\infty} e^{-t} P_b(n/m, zt) dt \tag{54}$$

otherwise. From there, it is apparent that the integrals for non-positive integral parameters do not converge as they would involve kernels of the form ${}_{n+1}F_0$ according to Equation (52). Therefore, the analytical continuation is defined only for negative integers a, b like in the case for the Wright function. In such case, (i.e., whenever $b = m \in \mathbb{N}$)

$$E_{-n,m}(z) = \int_0^{+\infty} e^{-t} P_m(n, zt) dt = 1 + \sum_{k=1}^{m} c_k z^k$$

which is a polynomial in z. The coefficients of this polynomial can be evaluated from the formula

$$c_k = a_k \int_0^{+\infty} e^{-t} t^k dt = k!\, a_k$$

Some examples can be presented as follows: For $n = 2$

$$1, z+1$$

For $n = 3$

$$\frac{1}{2}, \frac{2z+1}{2}, \frac{2z^2+2z+1}{2}$$

For $n = 4$

$$\frac{1}{6}, \frac{6z+1}{6}, \frac{6z+1}{6}, \frac{6z^3+6z^2+3z+1}{6}$$

For $n = 5$

$$\frac{1}{24}, \frac{24z+1}{24}, \frac{24z+1}{24}, \frac{24z^2+12z+1}{24}, \frac{24z^4+24z^3+12z^2+4z+1}{24},$$

etc. These polynomials can be rightfully called Mittag-Leffler polynomials.

11. Discussion

The original goal of the present work was to provide the ground truth for purely numerical algorithms for the evaluation of the Wright function. Such algorithms are a subject of continuous development [5,18,19].

The contributions of the present work can be discussed in several directions. In the first place, from a fundamental perspective, the existence of the Wright function of the third type has been overlooked in the literature. This can be probably attributed to the extant focus on Mainardi's function, which is not defined for $a = 1$. Moreover, the Mittag-Leffler function can be extended in a similar way. In the second place, the present work completes all cases of finite representations of the Wright function. It should be noted that the Second and Third Representation theorems could not be traced to the literature prior to the preliminary presentation in [10]. Finally, one can also envision an application in definite integration to be incorporated into different CAS integration—i.e., using Equation (52)—and Inverse Laplace transform routines—i.e., using Equations (46) and (47).

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Appendix A. Euler Integrals

The Gamma integral i.e., the Euler integral of the second kind is

$$\Gamma(z) = \int_0^{+\infty} e^{-\tau}\tau^{z-1}\, d\tau, \quad \mathrm{Re}\, z > 0 \tag{A1}$$

The complex representation for the reciprocal Gamma function is given by Heine's contour integral as

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{Ha^-} \frac{e^\tau}{\tau^z}\, d\tau \tag{A2}$$

Employing the last two formulas and the reflection formula of the Gamma function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z} \tag{A3}$$

one could obtain the analytical continuation of the Gamma function as valid on the entire complex plane for all $z \notin \mathbb{Z}$

$$\Gamma(z) = \frac{1}{2i \sin \pi z}\int_{Ha^-} e^\tau \tau^{z-1}\, d\tau, \quad \tau \in \mathbb{C} \tag{A4}$$

The Hankel contour is depicted in Figure 1. For non-integral arguments, the branch cut is selected as the negative real axis.

Appendix B. Ratios of Gamma Factors

Proposition A1. *For non-negative integers n, m*

$$\Gamma(mn + mb) = \Gamma(mb)\, m^{mn} \prod_{j=0}^{m-1} \left(\frac{j}{m} + b\right)_n \tag{A5}$$

Proof. Using the Gauss–Legendre multiplication formula for the Gamma function

$$\Gamma(mx) = \frac{m^{mx-1/2}}{(2\pi)^{(m-1)/2}} \prod_{k=0}^{m-1} \Gamma\left(x + \frac{k}{m}\right)$$

we first substitute $x = b$ then $x = n + b/m$ and divide the two identities. Thus, for a non-negative integer n, the formula can be expressed by a product of increasing factorials as

$$\frac{\Gamma(mn+mb)}{\Gamma(mb)} = m^{mn} \prod_{j=0}^{m-1} \frac{\Gamma\left(n+\frac{j}{m}+b\right)}{\Gamma\left(\frac{j}{m}+b\right)} = m^{mn} \prod_{j=0}^{m-1} \left(\frac{j}{m}+b\right)_n$$

□

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