On the Chromatic Number of the Plane with Two Forbidden Distances

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Abstract. The Chromatic Number of the Plane (CNP) problem can be generalized to the CNP with two forbidden distances. This problem seeks the smallest number of colors sufficient for coloring the Euclidean plane such that no two points at distance 1 or at distance d have the same color. A lower bound of six has previously been established for $d = \frac{1+\sqrt{5}}{2}$ and $d = 2$ by constructing a 6-chromatic 2-distance graph with edge lengths 1 and d. Here, we construct a 6-chromatic 2-distance graph for two new values, $d = \frac{\sqrt{2}+\sqrt{6}}{2}$ and $d = \sqrt{3}$. This way, the lower bound for the CNP with two forbidden distances 1 and d is raised to six, for the two new values of d. We then minimize the number of vertices in our 6-chromatic 2-distance graphs. For $d = \frac{\sqrt{2}+\sqrt{6}}{2}$ and $d = \sqrt{3}$, the smallest results are a 117-vertex graph and a 33-vertex graph respectively. The latter graph has only two vertices more than the smallest known 6-chromatic 2-distance graph. Both the verification of the 6-chromaticity and the graph minimization process were assisted by the use of a SAT solver.

1. INTRODUCTION. The Chromatic Number of the Plane (CNP), also denoted as $\chi(\mathbb{E}^2)$ or χ, is the smallest number of colors sufficient to color the Euclidean plane, such that no two points at unit distance have the same color. Though this formulation of the problem statement does not mention graphs, most of the proofs regarding the CNP make use of graphs. For a second definition of the CNP, using graph terminology, we introduce the concept of distance graphs.

Definition 1. A graph $G = (V, E)$, with vertex set V and edge set E, is a distance graph in n-dimensional Euclidean space with a distance set D if and only if conditions (1) and (2) are met.

(1) G has an injection $I : V \longrightarrow \mathbb{E}^n$ from its vertex set to the n-dimensional Euclidean space.

(2) G has a distance set $D \subset \mathbb{R}^+$ such that an edge between a vertex pair $\{u, v\}$ exists if and only if the distance between these vertices is in the distance set: $\forall u, v \in V : \{u, v\} \in E \iff d(I(u), I(v)) \in D$, with $d(x, y)$ being the Euclidean distance between two points x and y.

If $D = \{1\}$, G is called a unit-distance graph.

In this research, it is always assumed that a graph is embedded in the Euclidean plane $\mathbb{E}^2$. Since we only work with distance graphs and the injection implies that no two vertices can have the same position, a vertex is uniquely defined by its position. We leave the injection I behind and directly use $v = (v_x, v_y)$ to refer to a vertex v and its position alike.

Using graph terminology, the CNP is the chromatic number of the distance graph with all points of the Euclidean plane as its vertex set and edges between all vertex pairs at distance 1. The exact value of χ remains unknown to this day, but bounds have been set by previous authors.

Theorem 2 (Aubrey de Grey [1], John Isbell [2]). $5 \leq \chi \leq 7$

Edward Nelson is credited for first presenting the problem statement in 1950, though not in publication, together with a proof for a lower bound of four (see Alexander Soifer [2]). Generally, a lower bound k is proved by constructing a finite unit-distance graph with chromatic number k. Since the unit-distance graph with all points of the plane as its vertex set is a supergraph of every finite unit-distance graph, it is guaranteed that $\chi \geq k$. The Moser spindle is a 4-chromatic unit-distance graph with 7 vertices. Figure 1 shows a 4-coloring of the Moser spindle, with labels A/B/C/D representing the vertex colors. This graph serves as a proof for a lower bound of four for the CNP. For a long time, this was the best known lower bound. In 2018, de Grey [1] constructed a 5-chromatic unit-distance graph with 1581 vertices, raising the lower bound for the CNP to five. The long-standing upper bound is due to a 7-coloring of the plane with hexagonal tiles by Isbell (see Soifer [2]).

De Grey developed a custom program to verify the final step of his proof. In this work, we also use computer verification for our results. This is not done with a custom program, but with a satisfiability (SAT) solver. The use of SAT solvers, or use of computers for heavy calculations in general, has become increasingly popular and widely accepted in the context of mathematical proofs. The Four Color Theorem, a graph coloring problem that is much older than the CNP, is the first major theory that was proved with computer assistance in 1976 [3, 4]. Next to the CNP, Marijn Heule also used a SAT solver for other mathematical challenges, like his proof for the Boolean Pythagorean Triples Problem [5].

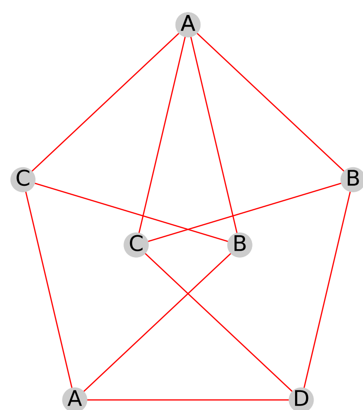


Figure 1. Moser spindle, the smallest 4-chromatic unit-distance graph.

The interest in the CNP was never gone and de Grey's revolutionary result increased the activity regarding this subject even further. Consequently, new proofs for $\chi \geq 5$ emerged. Jaan Parts [6] made a human-verifiable proof for $\chi \geq 5$. His proof is arguably a solution to, but certainly a big step towards a human proof, as opposed to the existing, computer-aided proofs. A polymath project [7] was launched to talk about ideas and new results regarding the CNP. This project entails a summary of important results on the CNP and some forum threads where enthusiasts can discuss their ideas. The main page is no longer online but the individual threads are still available.

Some aspects of the CNP can be generalized, resulting in related problems. One particular related problem is to generalize the forbidden distance 1 to a set D of forbidden distances. For the CNP with a distance set D, the notation χ_D is used. When this set contains two distances, $D = \{1, d\}$, the problem is called the CNP with two forbidden distances. No generality is lost by fixing the smallest of the two distances to 1, as the problem remains equivalent under scaling of the plane.

Again, a lower bound of k can be proved by constructing a k-chromatic graph. This time such a graph has edge lengths 1 and d, so it is called a 2-distance graph. A lower bound of five for any value of d is automatically implied by the existence of a 5-chromatic unit-distance graph. The question arises whether a lower bound of six can be proved by the construction of a 6-chromatic 2-distance graph for some values of d.

2. RELATED WORK AND KEY CONTRIBUTIONS. Matthew Huddleston [8] was the first to prove a lower bound of six for $d = \frac{1+\sqrt{5}}{2} =$ the golden ratio, though he never mentioned the CNP. In this proof, points of a pentagon are colored (without explicitly using the term graph), and with some combinatorial calculations, the non-5-colorability is proved. Parts [9] later revisited Huddleston's proof by creating the graph that was implicitly described by Huddleston in [8]. Parts succeeded in reducing the size of this graph from 251 vertices initially to 31 vertices finally. To this day this remains the smallest known 6-chromatic 2-distance graph. Geoffrey Exoo and Dan Ismailescu [10] constructed a 6-chromatic 2-distance graph with $d = 2$. This graph has 426 vertices and is less structured than Parts' graph.

An alternative way has been suggested for proving a lower bound on the CNP, also applicable to the CNP with two forbidden distances. Instead of constructing a k-chromatic distance graph to prove $\chi \geq k$, one can prove that there is a positive probability that a random graph exhibits the desired property of k-chromaticity. A so-called probabilistic proof then states that there must exist a graph with this property, even though it is not explicitly constructed. Paul Erdős [11] already used probabilistic methods in 1959, but the application to the CNP and its related problems only dates back to recent years.

Pálvölgyi Dömötör and Péter Ágoston presented a probabilistic proof for $\chi_{\{1,d\}} \geq 6$ for $d = \sqrt{3}$ and $d = \frac{\sqrt{2}+\sqrt{6}}{2}$. They give a high-level outline of this proof in the fourteenth Polymath thread [12]. Parts went further and probabilistically proved that $\chi_{\{1,d\}} \geq 7$ for $d = \frac{\sqrt{2}+\sqrt{6}}{2}$. He presents this in a few short steps in the fifteenth Polymath thread [13].

Despite proving an important result, none of these proofs have ever been published and neither have the propositions upon which the proofs are built. Although many details are omitted from the brief presentation of these probabilistic proofs, the core steps are presented convincingly and the peers of the authors give positive feedback in the subsequent comments. This speaks in favor of the validity of these results. However, even with full credit given to these proofs, they still only provide a lower bound on the CNP with two distances. With these proofs, the existence of a finite 6-chromatic two-distance graph for the distances $d = \sqrt{3}$ and $d = \frac{\sqrt{2}+\sqrt{6}}{2}$ might still depend on the axiom of choice, as mentioned by Dömötör [14]. Our research is the first to prove the existence of such graphs and construct instances of such graphs.

We construct a 6-chromatic 2-distance graph for the two values $d = \frac{\sqrt{2}+\sqrt{6}}{2}$ and $d = \sqrt{3}$. These values appeared in the probabilistic proofs mentioned above, but they are new for the construction of 6-chromatic 2-distance graphs. These two values of d are good candidates for a 6-chromatic graph because they allow for the complete graph K_4 to be embedded in the plane. When a graph contains two K_4 subgraphs of which three vertices are overlapping, the fourth vertices of both K_4 subgraphs form a pair that have the same color in any 4-coloring. This observation is the key to the similarity of these graphs to the Moser spindle, and therefore the possibility to obtain a proof similar to that of de Grey. For values other than $d = \frac{\sqrt{2}+\sqrt{6}}{2}$, $d = \sqrt{3}$, or $d = \frac{1+\sqrt{5}}{2}$, it is not possible to embed the complete graph K_4 in the plane in this way. One exception is $d = \sqrt{2}$. With this value, K_4 can be embedded, but not two K_4 subgraphs with three common vertices. With some increased symmetry, the graphs in Figure 2 and Figure 8 can be obtained for $d = \frac{\sqrt{2}+\sqrt{6}}{2}$ and $d = \sqrt{3}$, respectively. We will later call these graphs the S-graph (S for square) and the H-graph (H for hexagon). The similarity of these graphs to the Moser spindle (Figure 1) has not been described before. We are the first to investigate this similarity and exploit it in a construction that is based on de Grey's proof for $\chi \geq 5$. This process results in 6-chromatic 2-distance graphs, providing a constructive proof for Theorem 3 and Theorem 4.

Theorem 3. $\chi_{\{1, \frac{\sqrt{2}+\sqrt{6}}{2}\}} \geq 6$.

Theorem 4. $\chi_{\{1, \sqrt{3}\}} \geq 6$.

After constructing the first 6-chromatic 2-distance graphs for the new values of d, the secondary goal is to minimize the number of vertices in these graphs while maintaining the key property of 6-chromaticity, simply called graph minimization. Minimizing graphs is a whole research topic by itself within the field of the CNP. De Grey [1] already saw value in bringing his first 5-chromatic unit-distance graph down from 20425 vertices to 1581 vertices, even if the result was already revolutionary without the minimization. Heule [15] and Parts [16] specifically searched for ways to minimize 5-chromatic unit-distance graphs. Heule used a SAT solver to construct smaller graphs, while Parts worked directly on his graphs by repeatedly adding and deleting vertices. Their efforts led to the smallest known 5-chromatic unit-distance graph, standing at 509 vertices [16]. These two techniques will be combined in Section 4 when the minimization of our graphs is finalized.

The primary goal of this paper is to give constructive proofs of Theorem 3 and Theorem 4 by construction of a 6-chromatic 2-distance graph. First, Theorem 3 is proved in Section 3. The obtained 6-chromatic 2-distance graph is then minimized in Section 4. Finally, the whole procedure, including the minimization, is applied to 2-distance graphs with $d = \sqrt{3}$ in Section 5, leading to a proof for Theorem 4.

3. PROOF OF THEOREM 3.

S-spindle. For the construction of a 6-chromatic 2-distance graph, we use a method very similar to de Grey's method of constructing the first 5-chromatic unit-distance graph. We replace the Moser spindle, which plays a key role in de Grey's proof, with a similar graph, the S-spindle (described below). The subsequent steps follow the same reasoning as de Grey's proof and are only different in three places, resulting from the difference in geometry between the Moser spindle and the S-spindle.

We first consider the 5-chromatic 2-distance graph with $d = \frac{\sqrt{2}+\sqrt{6}}{2}$ in Figure 3. We call this graph the square-spindle or S-spindle, indicating that it is an extension of the Moser spindle to 2-distance graphs. The S-spindle consists of two rotated copies of the S-graph (Figure 2), each containing a small square and a rotated bigger square. In Figure 3, the left S-graph is indicated with thicker edges and black vertices, while the right S-graph is indicated with thinner edges and white vertices. The S-graphs are rotated around their lower vertex with an angle $\alpha = \arccos(\frac{3}{4})$, such that their upper vertices are at distance d and are thus connected by an edge (uppermost edge in Figure 3). Another observation is that the edges of length 1 in each S-graph make only angles that are a multiple of $\frac{\pi}{6}$, since the S-graph consists only of regular squares and triangles. Edges of length 1 are solid and red, while edges of length d are dashed and blue. These styles and colors for the edges will be used in all figures in this work.

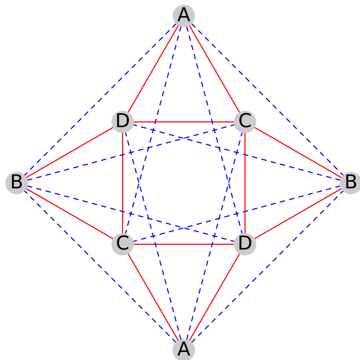


Figure 2. 4-coloring of the S-graph.

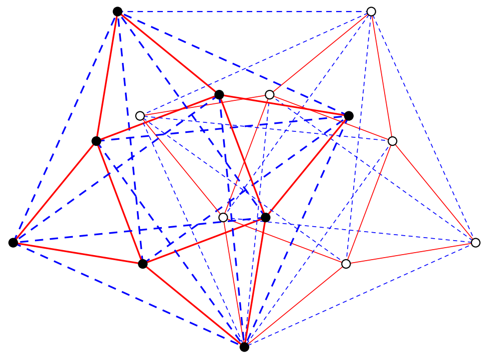


Figure 3. S-spindle, a small 5-chromatic 2-distance graph.

Proposition 5. *The S-spindle has chromatic number five.*

Proof. We try to find a 4-coloring of the S-spindle and conclude that at least five colors are needed by reaching a contradiction. The S-graph (Figure 2) contains eight vertices. Each vertex is connected with an edge to all other vertices except one, which is its mirrored image around the central symmetry point of the graph. Each unconnected vertex pair can thus be assigned one color, resulting in the only valid 4-coloring of this graph (Figure 2). The most important observation is that the upper and lower vertex always have the same color in a 4-coloring of the S-graph. This argument goes for both instances of the S-graph, meaning that the lower and both upper vertices of the S-spindle all share the same color in any 4-coloring. However, the angle between the S-graphs is chosen such that the upper vertices share an edge and cannot share the same color. This leads to a contradiction, proving that at least five colors are needed to color the S-spindle.

By the same construction as before, except that one of the upper vertices is colored with a fifth color, a valid 5-coloring is obtained. This proves the chromatic number of the S-spindle to be exactly five. ■

The S-spindle has the interesting property that it has a high chromatic number while having relatively few vertices. When constructing a 6-chromatic 2-distance graph, it is therefore a good idea to make this graph contain multiple copies of the S-spindle. If a graph contains more S-spindles that are interlocked by clever use of rotations and translations, the probability increases that no 5-coloring exists for this graph.

Graph V as a building block. Whereas the S-spindle serves as inspiration for our 6-chromatic graph, we now construct a graph V that can be directly used as a building block. The graph V uses two important elements from the S-spindle: the angle $\frac{\pi}{6}$ between the edges of length 1 within the S-graph and the angle $\alpha = \arccos(\frac{3}{4})$ between the two S-graphs. The graph V is now defined as the union of a vertex at the origin and 60 vertices on the unit circle at angles $\theta_{(i,j)} = i\frac{\pi}{6} + j\frac{\alpha}{2}, (i,j) \in [0..11] \times [-2..2]$ (Figure 4). The integer interval $[a..b]$ contains all integers between a and b, endpoints included: $[a..b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$.

Constructing a 6-chromatic 2-distance graph W. In this section we construct a non-5-colorable 2-distance graph W, using V as a building block. To this end, the Minkowski sum is introduced.

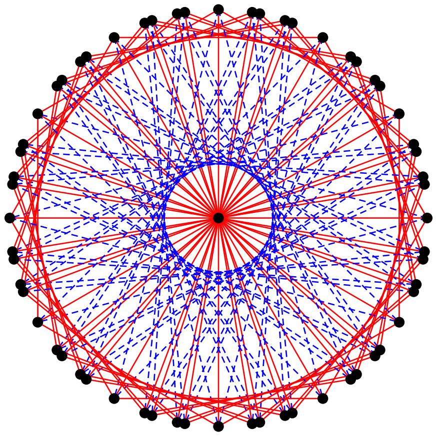


Figure 4. The 61-vertex graph V, the building block for 6-chromatic 2-distance graphs.

Definition 6. The Minkowski sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G_3 = (V_3, E_3)$ whose vertex set V_3 contains all component-wise additions of a vertex from V_1 and a vertex from V_2: $V_3 = \{w = (w_x, w_y) = (u_x + v_x, u_y + v_y) \mid \forall u \in V_1, v \in V_2\}$. The edge set E_3 is determined by the same distance set D as with graphs G_1 and G_2. The Minkowski sum is denoted as $G_3 = G_1 \oplus G_2$.

In terms of the Minkowski sum, W is the addition of three instances of V: $W = V \oplus V \oplus V$. A triple addition is needed because it takes three steps of unit length to go from the bottom vertex to the upper vertices in the S-spindle. With this construction, the resulting graph W contains many S-spindles, one of which is indicated in Figure 5.

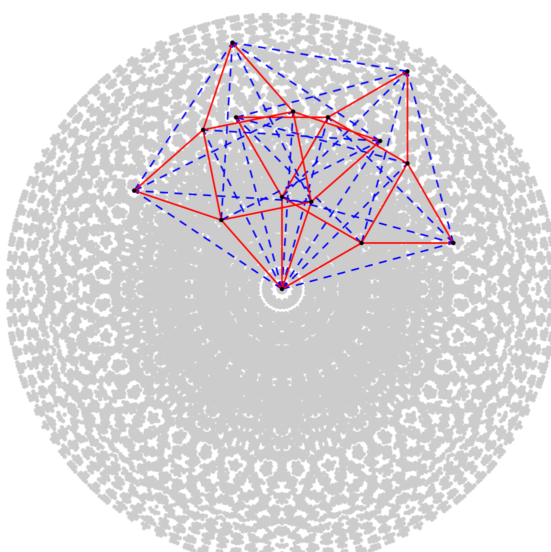


Figure 5. 6-chromatic 2-distance graph W with 34381 vertices and 310278 edges, with one S-spindle subgraph highlighted (edges are only shown for the S-spindle).

The graph W has 34381 vertices and 310278 edges. Its vertices are shown in Figure 5, though the high density makes it mostly visually impenetrable. To prove that no 5-coloring exists for this graph, we convert the graph coloring problem to a SAT problem. The details of this conversion are described in Section 4. We then use the SAT solver CP-SAT to prove the unsatisfiability of the SAT problem, meaning that no 5-coloring can be found for this graph. This is later also verified by other SAT solvers. As it is verified that the graph W has no valid 5-coloring, it serves as a constructive proof for Theorem 3.

Whether the chromatic number of W is six or higher is not determined, as the used graph coloring software does not terminate within a reasonable time when trying to color this graph with six colors. In the next section, when W is minimized, the smaller versions are always 6-chromatic.

4. MINIMIZING W. Like previous authors, we aim for a graph that has as few vertices as possible. This section considers a sequence of steps that reduce the number of vertices in W, while maintaining the property of non-5-colorability. The steps are divided into a manual part and an automatic part. The manual minimization decreases the size of W by systematically removing vertices from the constituent graph V. The automatic minimization removes vertices directly from W in a random fashion.

Manual minimization. As stated in Section 3, the 6-chromatic graph W is the triple Minkowski sum of V. This means that eliminating vertices from V results in many more vertices being eliminated from the final W. To create a graph V with fewer vertices, the angles of the vertices it has on the unit circle are reconsidered: $\theta_{(i,j)} = i\frac{\pi}{6} + j\frac{\alpha}{2}$. The full 61-vertex version of V uses this formula with a value set of $[0..11]$ for i and a value set of $[-2..2]$ for j. In the following, the value sets of both i and j are reduced in three steps. With a careful analysis of de Grey's construction and our construction of W, two important reduction steps are identified. These first two steps both aim to remove redundancies that were introduced by closely following de Grey's construction. Finally, the third step is on the border between the manual and automatic minimization, as it still operates on the constituent graph V but it makes use of an automated search for the optimal value sets.

Split V into even and odd rotations over $\frac{\pi}{6}$. As a first manual step, V is split into two versions V_{even} and V_{odd}, representing the even and odd rotations over $\frac{\pi}{6}$ in the formula for $\theta_{i,j}$: $\theta_{even,(i,j)} = i\frac{\pi}{3} + j\frac{\alpha}{2}, (i,j) \in [0..5] \times [-2..2]$ and $\theta_{odd,(i,j)} = \frac{\pi}{6} + i\frac{\pi}{3} + j\frac{\alpha}{2}, (i,j) \in [0..5] \times [-2..2]$. This distinction is not made in de Grey's construction, since the Moser spindle only contains the angle $\frac{\pi}{3}$ from the beginning. The shape of the S-spindle forced us to include angles that are a multiple of $\frac{\pi}{6}$, but no further distinction was initially made to keep as close as possible to de Grey's construction. The motivation to make this distinction now in the minimization lies in the nature of the three unit-distance edges that lead from the lower to the upper vertex in the S-spindle. The graph V_{even} contains two of these three edges and V_{odd} contains the other. This way, $W = V_{even} \oplus V_{even} \oplus V_{odd}$ still results in a graph containing many copies of the S-spindle. Note that $W^* = V_{even} \oplus V_{odd} \oplus V_{odd}$ would have resulted in the same final graph rotated over $\frac{\pi}{6}$ and therefore would have been an equally good choice. The resulting graph W now has 13981 vertices and 93822 edges. This graph is 6-chromatic as it is still non-5-colorable and the size now allows for the coloring software to find a 6-coloring. All graphs resulting from the next minimization steps will also be 6-chromatic.

Keep only one spindle angle. In this step, the value set of j is reduced to a minimum. Initially, four spindle angles, or rather four rational multiples of the spindle angle α were used: $\pm\alpha$ and $\pm\frac{\alpha}{2}$. This was because de Grey needed all these angles in his proof and we followed this in our construction. However, for our final graph W to contain many S-spindles, fewer spindle angles are sufficient. Keeping just one spindle angle suffices for the 6-chromaticity of the final graph W, corresponding to a value set for j of $\{-2, 0\}$ or $\{-1, 0\}$ after rescaling of the formula. This is a minimum that needs to be kept because a value set with one single value for j would make this variable obsolete and the important spindle angle α would not be found in the final graph anymore. Rewriting the formula for V_{even} and V_{odd}, new values for the angles on the unit circle are obtained: $\theta_{even,(i,j)} = i\frac{\pi}{3} + j\alpha, (i, j) \in [0..5] \times \{-1, 0\}$ and $\theta_{odd,(i,j)} = \frac{\pi}{6} + i\frac{\pi}{3} + j\alpha, (i, j) \in [0..5] \times \{-1, 0\}$. The resulting graph W has 949 vertices and 6912 edges.

Optimize value set of i. Now that we are working with graphs with fewer than 1000 vertices, a brute force optimization comes into reach for this final manual minimization step. To fully explore all optimization possibilities, a distinction is made between three versions of V: V_1, V_2, and V_3. The former two are subgraphs of V_{even}, the latter is a subgraph of V_{odd}. The value set of j is fixed at $\{-1, 0\}$ and the three different value sets of i are a subset of $[0..5]$. Out of all $(2^6)^3$ possible combinations of subsets, though this contains many isomorph cases, the three subsets are chosen which result in a final graph W with as few vertices as possible while still being 6-chromatic. A brute force calculation provides the optimal value sets of i: $\{0, 1, 2, 3\}$, $\{3, 4, 5\}$, and $\{0, 5\}$. This corresponds to graphs V with vertices on the unit circle at angles $\theta_{1,(i,j)} = i\frac{\pi}{3} + j\alpha, (i, j) \in [0..3] \times \{-1, 0\}$, $\theta_{2,(i,j)} = i\frac{\pi}{3} + j\alpha, (i, j) \in [3..5] \times \{-1, 0\}$, and $\theta_{3,(i,j)} = \frac{\pi}{6} + i\frac{\pi}{3} + j\alpha, (i, j) \in \{0, 5\} \times \{-1, 0\}$. Figure 6 shows the three versions of V corresponding to the optimal value sets. The resulting graph $W = V_1 \oplus V_2 \oplus V_3$ has 220 vertices and 1212 edges.

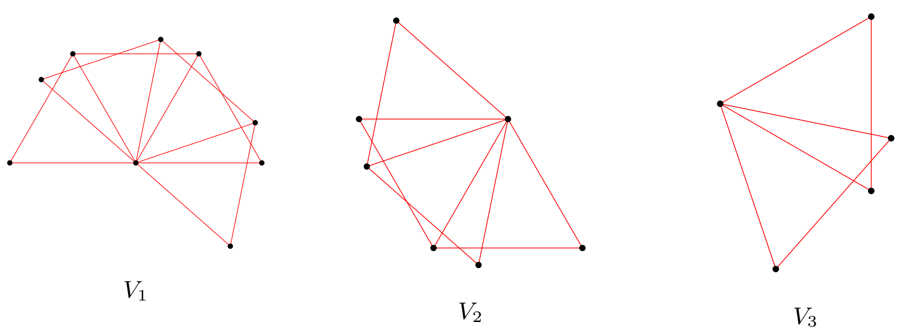


Figure 6. Graphs V_1, V_2, and V_3 which together form the smallest W.

Automatic minimization. The 6-chromatic graph that results from the manual minimization still has vertices that are non-critical, meaning that the graph is still 6-chromatic if this vertex is removed. The goal is to eliminate as many vertices as possible from W until only critical vertices remain.

We construct a short algorithm that combines methods from Heule and Parts. The algorithm alternates between a SAT solver and the direct deletion of vertices.

To make use of a SAT solver, the problem of coloring a graph $G = (V, E)$ with C colors must first be translated to a SAT problem. A SAT problem consists of variables, each with a set of possible values, and constraints on these variables. A SAT solver either assigns a value to each variable such that no constraint is violated (the problem is satisfiable) or it proves that such an assignment does not exist (the problem is unsatisfiable).

Each vertex v corresponds to C different boolean variables $x_{v,c}$, which represent the truth value of the statement "vertex v has color c". The vertex v also creates a constraint imposing that a vertex should at least have one color. Note that vertices are allowed to have multiple colors as this does not affect the satisfiability of the problem. Finally, an edge $\{u, v\}$ imposes C extra constraints $\neg(x_{u,c} \wedge x_{v,c})$. The meaning of these constraints is that for each color c, vertices u and v cannot both have this color if they share an edge. Equation (1) shows all constraints for the satisfiability problem that models a graph coloring problem. The vertex set of the graph is denoted as V, the edge set as E, and the number of colors used for coloring the graph as C.

$$\bigwedge_{v\in V} \bigvee_{c=0}^{C-1} x_{v,c} \wedge \bigwedge_{(u,v)\in E} \bigwedge_{c=0}^{C-1} \neg(x_{u,c} \wedge x_{v,c}) \tag{1}$$

A SAT solver returns whether a valid assignment exists for all variables, where none of the constraints is violated. In our case, this corresponds to a valid 5-coloring of the graph. Moreover, the SAT solver can also extract an unsatisfiable core from the given constraints in the case of unsatisfiability. Such an unsatisfiable core is a subset of the initial constraints which is sufficient to prove unsatisfiability, leaving out some constraints that are not needed for the unsatisfiability. If all constraints corresponding to a vertex and its edges are left out of the core, this vertex can be eliminated from the graph without losing the non-5-colorability.

Eliminating vertices with a SAT solver has the advantage that multiple vertices can be removed at once. The SAT solver is not perfect, so it can get stuck on a certain subgraph that still has non-critical vertices. At this point, Heule decides to change his method and he starts directly eliminating non-critical vertices until only critical vertices remain. We alter Heule's method in such a way that a better balance is acquired between the SAT solver and vertex elimination. We noticed that the SAT solver could still provide smaller unsatisfiable cores after manually deleting one vertex from a graph on which the SAT solver was previously stuck. This way, we extend the possibilities of the SAT solver beyond the graphs on which it initially gets stuck.

It is important to keep a balance when alternating between the two methods. The SAT solver is mostly successful at the start of the algorithm when the graph is still big. When the graph nears its minimal form, only the direct vertex elimination can bring improvements. This balance is dynamically maintained until we reach a graph with only critical vertices. Due to the random nature of both the SAT solver and the vertex selection, multiple iterations are executed, each time resulting in a slightly different graph. This procedure eventually results in a graph with 117 vertices and 565 edges (Figure 7). This 6-chromatic 2-distance graph has considerably fewer vertices than the graph of Exoo and Ismailescu, but still a lot more than the graph of Parts.

5. PROOF OF THEOREM 4 AND MINIMIZATION. We only needed minor changes for de Grey's method to be successfully applicable to 6-chromatic 2-distance graphs with $d = \frac{\sqrt{2}+\sqrt{6}}{2}$. It turns out that the method is even more powerful for the case with $d = \sqrt{3}$. Almost all steps are identical to those for $d = \frac{\sqrt{2}+\sqrt{6}}{2}$, both for the construction and the minimization, so we only focus on the three points where there is

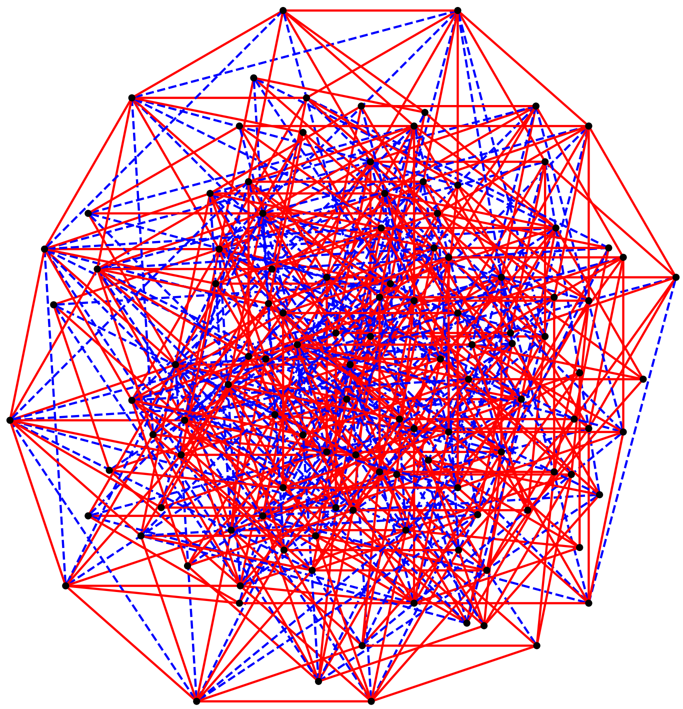


Figure 7. The smallest 6-chromatic 2-distance graph for $d = \frac{\sqrt{2}+\sqrt{6}}{2}$, having 117 vertices and 565 edges.

a difference.

First, the hexagon spindle or H-spindle (Figure 9) serves as an alternative to the S-spindle. The construction of the H-spindle is completely similar to that of the S-spindle, except that now two copies of the H-graph are used instead of two copies of the S-graph and the spindle angle $\alpha = \arccos(\frac{7}{8})$ is different. In Figure 8, the H-graph is drawn with solid, red edges of length 1 and dashed, blue edges of length d.

A second difference also follows from the difference between the S-graph and the H-graph. In the first step of manual minimization, V was previously split into even and odd rotations over $\frac{\pi}{6}$ because some unit-length edges in the S-graph make an angle that is an odd multiple of $\frac{\pi}{6}$. Since all edges in the H-graph make angles that are a multiple of $\frac{\pi}{3}$, this split between two versions of V is no longer necessary. In other words, only one formula for the angles of the vertices of V on the unit circle is needed: $\theta_{(i,j)} = i\frac{\pi}{3} + j\frac{\alpha}{2}, (i,j) \in [0..5] \times [-2..2]$. It is now only in the final manual minimization step that different versions of V are used.

In this final manual minimization step, the optimal value sets for i in the formula of three different versions of V are searched by brute force. The third difference lies in these value sets: $\{0, 1, 2, 3, 4\}$, $\{3\}$, and $\{1\}$ now result in the smallest graph $W = V_1 \oplus V_2 \oplus V_3$ with 58 vertices and 263 edges. The graphs V_1, V_2, and V_3 are shown in Figure 10, where it can be seen that only three of the initial 61 vertices are left in the graphs V_2 and V_3.

From here on, the automatic minimization starts again. The randomness of the minimization algorithm is now less important due to the smaller number of nodes. The

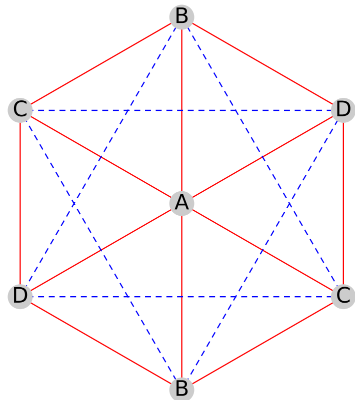


Figure 8. 4-coloring of the H-graph.

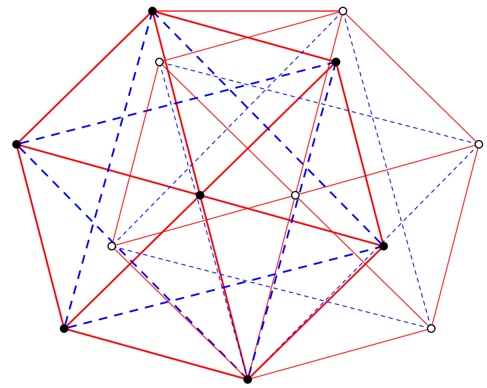
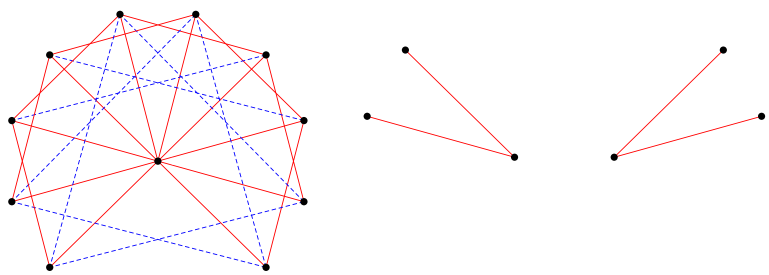


Figure 9. H-spindle, a small 5-chromatic 2-distance graph.



V_1 V_2 V_3

Figure 10. Graphs V_1, V_2, and V_3 which together form the smallest W for $d = \sqrt{3}$.

resulting minimal graphs mostly had 36 vertices. We escape this local minimum by making the initial graph bigger.

Mirroring the graph W obtained by the manual minimization around the horizontal line $y = \frac{\sqrt{15}}{4}$, adds 18 new vertices to W. If we use this new 76-vertex W as input for the automatic minimization, smaller graphs are found. The best result is a graph with 33 vertices and 113 edges (Figure 11). This is the smallest 6-chromatic 2-distance graph in this research, having only 2 vertices more than the smallest known 6-chromatic 2-distance graph by Parts [9].

6. CONCLUSIONS AND FUTURE WORK. In this work, we presented a constructive proof for a lower bound of six for the CNP with two forbidden distances, 1 and d. Such a proof is given by construction of a 6-chromatic 2-distance graph. Previous authors had obtained a 6-chromatic 2-distance graph for $d = \frac{1+\sqrt{5}}{2}$ and $d = 2$. We provided a proof for two new values, $d = \frac{\sqrt{2}+\sqrt{6}}{2}$ and $d = \sqrt{3}$. The method of constructing these new graphs was inspired by de Grey's construction of the first 5-chromatic unit-distance graph. This work shows that the application of de Grey's method can be successful for values of d that allow the construction of small graphs showing high similarity with the Moser spindle (Figure 1), which plays a key role in

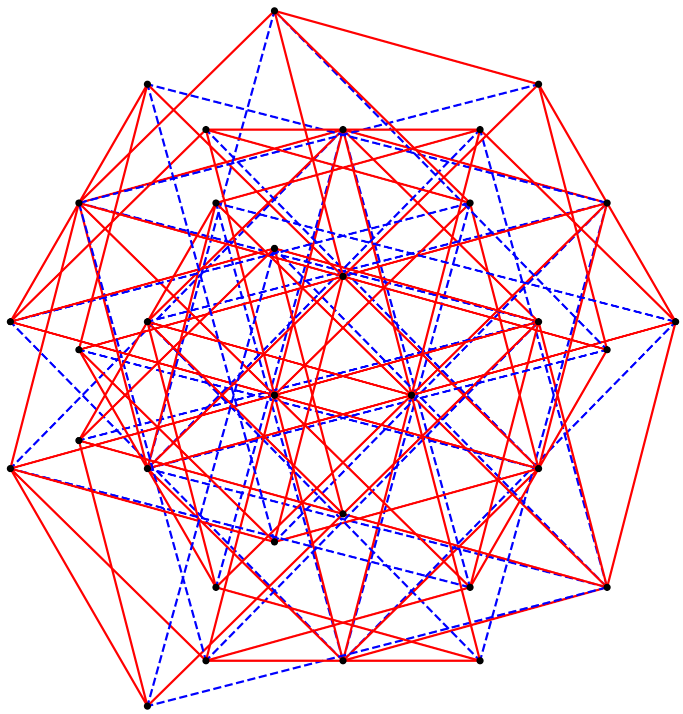


Figure 11. The smallest 6-chromatic 2-distance graph for $d = \sqrt{3}$, having 33 vertices and 113 edges.

de Grey's proof. The described methods would be harder to apply to other values of d because it is not possible to obtain these similar graphs. The question whether these methods can still be applied for other values of d remains interesting, for example for those distances that allow 5-chromatic 2-distance graphs with relatively few vertices to be embedded, even if these graphs are less similar to the Moser spindle. The question also arises whether de Grey's method can be used for more problems related to the CNP. For example, can this method be used to increase the lower bound of six for the Chromatic Number of Space, a related problem where points in Euclidean space ($\mathbb{E}^3$) are considered rather than the Euclidean plane ($\mathbb{E}^2$)? The Moser spindle can be extended to a three-dimensional Moser spindle, so with some tweaking a similar extension might be found for the other steps in the construction.

This work is the first to prove a new lower bound and immediately perform a thorough minimization on the obtained graphs. For the graphs in previous work, the minimization was done by a different author (for $d = \frac{1+\sqrt{5}}{2}$) or not at all (for $d = 2$). During the automatic minimization, we introduced a new procedure to remove vertices from a graph. This procedure alternated between using a SAT solver and directly removing vertices, dynamically balancing the strengths of both methods. Our minimization process resulted in graphs with 117 vertices and 33 vertices for $d = \frac{\sqrt{2}+\sqrt{6}}{2}$ and $d = \sqrt{3}$ respectively. The latter graph is only two vertices short of equalizing the record for the smallest known 6-chromatic 2-distance graph. The valuable graphs that resulted from the minimization procedure show how the use of SAT solvers and computers in general can greatly assist advances in mathematics, certainly in topics involving large graphs or other computation-intensive topics. Much of the recent research regarding the CNP has had a strong computational component and could therefore not have been conducted without computer assistance, and this is not likely to decline in future research.

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Appendix: Vertex coordinates of smallest graphs. The smallest versions of the graph W, for both $d = \frac{\sqrt{2}+\sqrt{6}}{2}$ and $d = \sqrt{3}$, result from random steps in the automatic minimization. Until now, the final graphs were only presented graphically in Figure 7 and Figure 11. This appendix provides tables with the exact coordinates of each vertex, guaranteeing the reproducibility of our results.

Due to the specific way of constructing our graphs, all coordinates lie in an extension field of the rational numbers. For $d = \frac{\sqrt{2}+\sqrt{6}}{2}$, this extension field is $\mathbb{Q}(\sqrt{3}, \sqrt{7}) = \{a_0 + a_1\sqrt{3} + a_2\sqrt{7} + a_3\sqrt{21} \mid a_i \in \mathbb{Q}\}$. More specifically, all vertex coordinates can be written as $(v_x, v_y) = (\frac{a_0+a_1\sqrt{3}+a_2\sqrt{7}+a_3\sqrt{21}}{8}, \frac{a_4+a_5\sqrt{3}+a_6\sqrt{7}+a_7\sqrt{21}}{8})$ with $a_i \in \mathbb{Z}$. Table 2 shows coordinates of all 117 vertices in the minimal graph W, using the condensed notation $[a_0\ a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7]$.

For $d = \sqrt{3}$, the extension field is $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ and all coordinates can be written as $(v_x, v_y) = (\frac{a_0+a_1\sqrt{5}}{8}, \frac{a_2\sqrt{3}+a_3\sqrt{15}}{8})$ with $a_i \in \mathbb{Z}$. This allows for an even denser notation $[a_0\ a_1\ a_2\ a_3]$, shown in Table 1 for all 33 vertices of the minimal graph W.

The edges are not specified for these graphs as they can be calculated using the vertex coordinates and the distance set $D = \{1, d\}$.

Table 1. Vertex coordinates of the smallest graph W for $d = \sqrt{3}$.

[0 0 0 0]	[0 0 0 4]	[1 3 -1 1]	[1 3 1 3]	[-1 3 1 1]
[-1 3 -1 3]	[1 -3 1 1]	[1 -3 -1 3]	[2 0 0 2]	[0 0 2 2]
[0 0 -2 2]	[-1 -3 1 3]	[-2 0 0 2]	[-2 0 2 4]	[-1 3 3 3]
[-3 3 -1 1]	[-3 3 1 3]	[3 3 -1 3]	[1 3 3 1]	[3 -3 1 3]
[3 -3 -1 1]	[4 0 0 4]	[4 0 0 0]	[-3 -3 -1 3]	[-3 -3 1 1]
[-2 0 -2 4]	[-2 0 2 0]	[-4 0 0 4]	[-4 0 0 0]	[1 -3 3 3]
[1 -3 -3 1]	[-1 -3 -3 3]	[-1 -3 3 1]		

Table 2. Vertex coordinates of the smallest graph W for $d = \frac{\sqrt{2}+\sqrt{6}}{2}$.

[0 0 0 0 0 0 0 0]	[0 3 1 0 3 0 0 -1]	[0 4 0 0 4 0 0 0]
[0 3 -1 0 -3 0 0 -1]	[0 4 0 0 -4 0 0 0]	[-6 0 0 0 0 0 2 0]
[-6 3 1 0 3 0 2 -1]	[-6 4 0 0 4 0 2 0]	[-6 3 -1 0 -3 0 2 -1]
[-6 4 0 0 -4 0 2 0]	[-8 0 0 0 0 0 0 0]	[-8 3 1 0 3 0 0 -1]
[-8 4 0 0 4 0 0 0]	[-8 3 -1 0 -3 0 0 -1]	[-8 4 0 0 -4 0 0 0]
[-3 0 0 -1 0 -3 1 0]	[-3 3 1 -1 3 -3 1 -1]	[-3 4 0 -1 4 -3 1 0]
[-3 3 -1 -1 -3 -3 1 -1]	[-3 4 0 -1 -4 -3 1 0]	[-4 0 0 0 0 -4 0 0]
[-4 4 0 0 4 -4 0 0]	[-4 3 -1 0 -3 -4 0 -1]	[-4 4 0 0 -4 -4 0 0]
[3 0 0 -1 0 -3 -1 0]	[3 3 1 -1 3 -3 -1 -1]	[3 3 -1 -1 -3 -3 -1 -1]
[3 4 0 -1 -4 -3 -1 0]	[4 4 0 0 4 -4 0 0]	[4 4 0 0 -4 -4 0 0]
[6 3 -1 0 -3 0 -2 -1]	[-2 3 1 0 3 0 -2 -1]	[-2 3 -1 0 -3 0 -2 -1]
[8 4 0 0 4 0 0 0]	[2 4 0 0 4 0 2 0]	[2 4 0 0 -4 0 2 0]
[5 4 0 -1 4 -3 1 0]	[3 3 -1 1 -3 3 -1 -1]	[-3 0 0 1 0 3 1 0]
[-3 3 1 1 3 3 1 -1]	[-3 4 0 1 4 3 1 0]	[-3 3 -1 1 -3 3 1 -1]
[-5 3 1 1 3 3 -1 -1]	[-5 3 -1 1 -3 3 -1 -1]	[-1 0 0 1 0 -1 -1 0]
[-1 3 1 1 3 -1 -1 -1]	[-1 4 0 1 4 -1 -1 0]	[-1 3 -1 1 -3 -1 -1 -1]
[-1 4 0 1 -4 -1 -1 0]	[4 0 0 0 0 4 0 0]	[4 3 1 0 3 4 0 -1]
[4 4 0 0 4 4 0 0]	[4 4 0 0 -4 4 0 0]	[-2 0 0 0 0 4 2 0]
[-2 4 0 0 4 4 2 0]	[-2 4 0 0 -4 4 2 0]	[-4 0 0 0 0 4 0 0]
[-4 3 1 0 3 4 0 -1]	[-4 4 0 0 4 4 0 0]	[-4 4 0 0 -4 4 0 0]
[1 0 0 -1 0 1 1 0]	[1 4 0 -1 4 1 1 0]	[7 3 1 -1 3 1 -1 -1]
[7 4 0 -1 4 1 -1 0]	[7 3 -1 -1 -3 1 -1 -1]	[7 4 0 -1 -4 1 -1 0]
[-9 0 0 1 0 3 3 0]	[-9 3 1 1 3 3 3 -1]	[-11 0 0 1 0 3 1 0]
[-11 3 1 1 3 3 1 -1]	[-11 4 0 1 4 3 1 0]	[-11 3 -1 1 -3 3 1 -1]
[-7 0 0 1 0 -1 1 0]	[-7 3 1 1 3 -1 1 -1]	[-7 4 0 1 4 -1 1 0]
[-7 3 -1 1 -3 -1 1 -1]	[-7 4 0 1 -4 -1 1 0]	[1 0 0 1 0 -1 1 0]
[1 3 1 1 3 -1 1 -1]	[1 4 0 1 4 -1 1 0]	[1 3 -1 1 -3 -1 1 -1]
[1 4 0 1 -4 -1 1 0]	[-10 0 0 0 0 4 2 0]	[-10 3 1 0 3 4 2 -1]
[-10 4 0 0 4 4 2 0]	[-12 0 0 0 0 4 0 0]	[-12 4 0 0 4 4 0 0]
[-7 0 0 -1 0 1 1 0]	[-7 3 1 -1 3 1 1 -1]	[-7 4 0 -1 4 1 1 0]
[-7 3 -1 -1 -3 1 1 -1]	[-7 4 0 -1 -4 1 1 0]	[-1 0 0 -1 0 1 -1 0]
[-1 3 1 -1 3 1 -1 -1]	[-1 4 0 -1 4 1 -1 0]	[-1 3 -1 -1 -3 1 -1 -1]
[-1 4 0 -1 -4 1 -1 0]	[-14 0 0 0 0 0 2 0]	[-14 3 1 0 3 0 2 -1]
[-9 0 0 -1 0 -3 3 0]	[-9 3 -1 -1 -3 -3 3 -1]	[-10 0 0 0 0 -4 2 0]
[-10 3 1 0 3 -4 2 -1]	[-10 4 0 0 4 -4 2 0]	[-10 3 -1 0 -3 -4 2 -1]
[-10 4 0 0 -4 -4 2 0]	[-2 4 0 0 4 -4 2 0]	[-2 4 0 0 -4 -4 2 0]
[-11 0 0 -1 0 -3 1 0]	[-11 3 1 -1 3 -3 1 -1]	[-11 3 -1 -1 -3 -3 1 -1]
[-11 4 0 -1 -4 -3 1 0]	[-12 0 0 0 0 -4 0 0]	[-12 4 0 0 -4 -4 0 0]
[-5 0 0 -1 0 -3 -1 0]	[-5 3 1 -1 3 -3 -1 -1]	[-5 3 -1 -1 -3 -3 -1 -1]