

Analytic Solution and Spectral Analysis of the Differential Surface Admittance Electric Field Integral Equation for Spherical Geometries

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Abstract—In order to enable a rigorous spectral analysis of the existing single-source differential surface admittance electric field integral equation (DSA-EFIE), a fully analytical solution for scattering at a nonmagnetic homogeneous dielectric sphere is presented. To this end, the pertinent surface unknowns and operators are expanded into vector spherical harmonics, and subjected to a Galerkin method of moments. The resulting integrals are all computed analytically, the solution is proven to be fully equivalent to the reference Mie series and the spectral properties are derived for any Sobolev testing space.

Consequently several important insights are rigorously shown in the spectral analysis. First and foremost, the DSA-EFIE provides an exact solution of Maxwell's equations for any complex dielectric material and frequency, establishing the approach as a precise single-source boundary integral formulation. Moreover, through the application of the generalized Fourier series, closed analytical formulas are derived for all pertinent elements, leading to expressions devoid of any numerical integration or summation, thereby strengthening the method's efficacy and accuracy. Additionally, this work shows that the concatenation of the two pertinent integral operators provides a formulation with a bounded condition number in both the space of square integrable functions and the Sobolev $H_{\text{div}}^{-1/2}$ space, required for a bounded EFIE Galerkin solution. Moreover, it is deduced that the inherent low-frequency breakdown of the EFIE is avoided as well. Lastly, the issue of internal resonances, typical for single-source formulations, persist although they are not simply those of the DSA and EFIE operator on their own. Hence, the advocated analysis approach has brought forth important insights and provides a useful avenue for analyzing alternative integral equation formulations with extended capabilities or improved properties.

Index Terms—Analytic solution, integral equations, method of moments (MoM), single source, differential surface admittance (DSA) operator.

I. INTRODUCTION

SPURRED by the ever-growing demand for more powerful and versatile communication systems and electronic devices, circuit designers are faced with enormous challenges, such as miniaturization, the inclusion of novel materials [1],

and the heterogeneous integration of various technology platforms [2]. Moreover, the rising operating frequency strengthens the impact of full-wave effects, such as crosstalk, distortion and frequency-dependent losses, which require rigorous computational electromagnetic methods (CEMs).

Boundary integral equations [3] (BIEs) are a broad class of CEMs that rely on the equivalence principle [4] and the Green's function [5] to restrict the unknowns to the boundary interfaces. BIEs are traditionally formulated by introducing two sources on the boundary, leading to various established methods [6]–[11]. However, the equivalence principle can be applied with one equivalent source as well [12]. By giving up full control over the field distribution on the inside of the replaced medium, as in traditional double-source methods, only one unknown boundary quantity is required for a full description of the problem outside the replaced medium [13]. This leads to the so-called single-source integral equations, of which many incarnations have been developed [14]–[18]. Note that, if so desired, the actual fields inside the original medium can be recuperated in a post-processing step once the boundary quantities are calculated.

One such approach leverages the differential surface admittance operator (DSA) [19], which introduces only an electric surface current density on the boundary while preserving the field quantities on the outside. This (fictitious) current density is consequently expressed as an operator acting on the boundary electric field and can be computed through the volume's eigenmodes [19], the Fokas method [20] or via the conventional BIE operators [21]. First established for 2-D rectangular and circular cross-sections [19], [22], it has been expanded to other shapes such as triangles and polygons [23], to include semiconductors and magnetic materials [24], and to 3-D equivalents for cylinders and cuboids [25]–[27]. Throughout these publications, the DSA approach has demonstrated broadband numerical accuracy, competitive computation times, and diverse application opportunities. Nevertheless, the fictitious nature of the introduced current density and the fact that the field distribution inside the object is not reconstructed explicitly, continues to raise points of concern on its inherent rigor, and by extension on its convergence properties and frequency-dependent behavior.

To meticulously validate the rigorous nature of the DSA operator, in this paper we establish the exact, analytical solution for a single, homogeneous, dielectric sphere for the BIE system consisting of the DSA operator, introduced through the

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equivalence theorem, and the electric field integral equation (EFIE) to solve the exterior problem. Through the use of a Galerkin method of moments (MoM) calculation, inspired by the procedure and naming convention outlined in [7], [28]–[30] for the EFIE and the Surface-Volume-Surface-EFIE (SVS-EFIE), the unknown surface quantities are expressed in a spherical basis. By leveraging vector spherical harmonics and their orthogonality, all relevant inner products are computed fully analytically for any Sobolev testing space H_{div}^l , leading to closed expressions. A crucial step in the computation of the DSA discretization is the employment of the generalized Fourier series [31], which enabled us to find closed sums for slowly convergent series, resulting in a fast and accurate evaluation without any need for numerical integration or truncated summations. The first steps in this derivation have been published before [32] [33], where we focused solely on the excitation by a radial and tangential dipole, respectively. In this paper, we perform the analysis without any constraints on the excitation, hence leading to an original, complete description of the DSA-EFIE system for a spherical geometry. Moreover, we generalize the discretization procedure to take into account different Sobolev testing spaces H_{div}^l , enabling an in-depth study of the spectral properties of the DSA-EFIE system for the first time. Additionally, we proof the mathematical equivalence to the reference result, i.e., the Mie series, to underscore the correctness of the DSA formulation and illustrate the derived spectral properties.

The theory and proof clearly put five points into perspective. First, the DSA-EFIE does indeed rigorously solves Maxwell's equations and hence it provides an accurate solution, despite its unconventional formulation. Second, the closed sums for the DSA calculations enhance the applicability for all dielectric materials, in particular good conductors, which typically require special care/routines [34]. Third, the simple concatenation of the DSA & EFIE is shown to be free from dense-mesh breakdown. Moreover, it is revealed that this is the case in both the convenient $L^2(S)$ space of square integrable functions as well as the Sobolev space $H_{\text{div}}^{-1/2}(S)$, which is typically required for a bounded solution of the EFIE on its own [28], as the two sets of eigenmodes are shown to accumulate to a finite value, leading to a bounded condition number. Fourth, the EFIE's inherent low-frequency breakdown is solved by the inclusion of the DSA operator since two finite accumulation points emerge for the sets of eigenvalues irregardless of the testing space. Finally, as a single-source integral equation approach, the DSA-EFIE suffers from internal resonances proven to be introduced by the operators and their product.

Although the distinct DSA varieties have been widely applied to a plethora of applications, the results summarized above have never been backed by a rigorously derived analytical proof before. They show that observed breakdowns of numerical DSA-EFIE formulation are not inherent issues that can thus be solved by selecting a correct testing space, or avoiding other sources of ill-conditioning, just like it is the case with other meticulously investigated BIEs such as the EFIE [28] or the SVS-EFIE [29] [30]. This newly developed mathematical framework provides an excellent and insightful

toolbox to explore and construct alternative formulations of this integral equation with improved properties or extended capabilities. One interesting avenue that could be explored in the future is the impact of the discretization strategy on the spectral properties. This analysis was recently performed for established 2-D integral operators in the context of high-frequency breakdown [35], [36], with the latest extensions to non-circular geometries and 3-D objects [37], [38] particularly relevant to the work presented in this paper.

Section II briefly revisits the application of the single-source equivalence theorem, and the EFIE and DSA operator that are utilized to solve the resulting equivalent problem. The discretization procedure for a spherical geometry is then described in Section III, including the impact of the choice of the Sobolev testing space on the inner product expressions. In Section IV, the use of the generalized Fourier series to construct closed form expressions for the discretized DSA elements is outlined, followed by the impact of various breakdown scenarios on the system's eigenvalues. The mathematical proof that the DSA-EFIE solution and the classical Mie series result in exactly the same field expressions is outlined in Section V. Afterwards, we illustrate the impact of the closed sums for the infinite DSA series and the spectral behavior of the operators. Finally, conclusions are formulated in Section VII.

II. SINGLE-SOURCE INTEGRAL OPERATORS

Consider the situation visualized in Figure 1a: a nonmagnetic, dielectric object $\mathcal{V}$ with permittivity ϵ is situated in a background medium $\mathcal{V}_0$, e.g., free space, bounded by its surface $\mathcal{S}$. In accordance with the single-source equivalence theorem [12], the material inside can be switched out for the background medium by introducing an electric surface current density $\mathbf{j}_s$ on the boundary. As shown in Figure 1b, by the correct choice of this current density, we can ensure that the fields outside, viz., $\mathbf{e}_0$ and $\mathbf{h}_0$, are preserved while the $\mathbf{e}$ and $\mathbf{h}$ fields inside the material are replaced by the fictitious fields $\mathbf{e}'$ and $\mathbf{h}'$, respectively. With the material discontinuity removed, the fields generated by the as of yet unknown $\mathbf{j}_s$ can be expressed by a standard electric field integral equation (EFIE) formulation on the boundary $\mathcal{S}$:

$$\hat{\mathbf{n}} \times \mathbf{e}_0^t = \hat{\mathbf{n}} \times \mathbf{e}_{\text{inc}} + \mathcal{T} \circ \mathbf{j}_s, \quad (1)$$

with $\mathbf{e}_{\text{inc}}$ the incident electric field generated by sources in $\mathcal{V}_0$, the superscript t emphasizing the tangential nature of the field component, and the operator $\mathcal{T}$, also known as the single-layer operator, defined as

$$\mathcal{T} \circ \mathbf{a} = -j\omega\mu_0 \hat{\mathbf{n}} \times \int_{\mathcal{S}} \bar{\bar{\mathbf{G}}}_0(|\mathbf{r} - \mathbf{r}'|) \mathbf{a}(\mathbf{r}') d\mathbf{r}', \quad (2)$$

where, adopting the time-harmonic $e^{j\omega t}$ convention, $k_0 = \omega\sqrt{\epsilon_0\mu_0}$ and the free-space Green's dyadic is given by

$$\bar{\bar{\mathbf{G}}}_0(|\mathbf{r} - \mathbf{r}'|) = \left(\bar{\bar{\mathbf{I}}} + \frac{\nabla\nabla}{k_0^2} \right) \frac{e^{-jk_0|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}. \quad (3)$$

Multiple approaches exist to find an expression for $\mathbf{j}_s$. Here we first obtain a general formulation by introducing two Poincaré-Steklov (PS) operators [39], or more specifically Dirichlet-to-Neumann (DtN) operators, that map the tangential electric

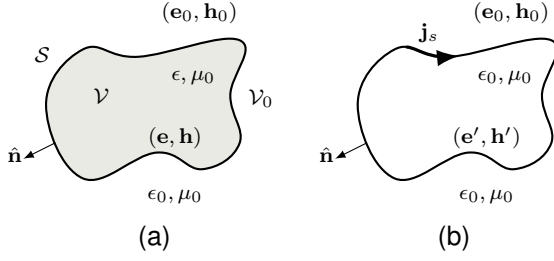


Fig. 1. Illustration of the single-source equivalence principle for a general geometry with (a) the original set-up and (b) the equivalent situation where the medium inside $\mathcal{V}$ has been replaced by the background material by imposing $\mathbf{j}_s$.

field on the boundary just inside $\mathcal{V}$ to its rotated magnetic counterpart:

$$\hat{\mathbf{n}} \times \mathbf{h} = \mathcal{P} \circ \mathbf{e}^t \quad \hat{\mathbf{n}} \times \mathbf{h}' = \mathcal{P}' \circ \mathbf{e}'^t, \quad (4)$$

where the two operators apply to the scenarios visualized in Figure 1a and Figure 1b, respectively. Taking the boundary condition on the tangential electric field into account, one easily sees that $\mathbf{e}^t = \mathbf{e}'^t = \mathbf{e}_0^t$. Subtracting the tangential magnetic field in both situations and introducing (4), leads to an expression for $\mathbf{j}_s$ as a function of $\mathbf{e}_0^t$:

$$\mathbf{j}_s = \hat{\mathbf{n}} \times (\mathbf{h} - \mathbf{h}') = (\mathcal{P} - \mathcal{P}') \circ \mathbf{e}_0^t \quad (5)$$

$$= \mathcal{Y} \circ \mathbf{e}_0^t, \quad (6)$$

where we introduce the general form of the differential surface admittance (DSA) operator $\mathcal{Y}$. These PS operators, and hence their difference $\mathcal{Y}$, can be expressed in terms of a boundary integral equation [21], [40] or equivalently, through the electric and magnetic integral operators $\mathcal{T}$ and $\mathcal{K}$ [41]. However, these approaches make use of the Green's function in the original material occupying $\mathcal{V}$. Hence, we focus here on the original approach [19] and its extension to 3-D [26] where the DSA operator is instead constructed based on the magnetic Dirichlet eigenfunctions of a PEC cavity with the same form as $\mathcal{V}$, leading to the following form of $\mathcal{Y}$ [25]:

$$\mathbf{j}_s = -\frac{k^2 - k_0^2}{j\omega\mu_0} \times \sum_{\nu} \frac{k_{\nu}^2 (\hat{\mathbf{n}} \times \mathbf{h}_{\nu})}{(k_{\nu}^2 - k^2)(k_{\nu}^2 - k_0^2)\mathcal{N}_{\nu}^2} \int_S (\hat{\mathbf{n}} \times \mathbf{h}_{\nu}^*) \cdot \mathbf{e}_0^t dS, \quad (7)$$

with $k = \omega\sqrt{\epsilon\mu_0}$ the wavenumber in the material, $\mathbf{h}_{\nu}$ a magnetic eigenfunction of $\mathcal{V}$, k_{ν} and $\mathcal{N}_{\nu}$ its wavenumber and normalization constant, respectively, and ν a triple index covering all eigenfunctions. A full derivation of this expression is found in [26].

III. EXACT GALERKIN SOLUTION OF THE DSA-EFIE ON A SPHERE

To investigate the fundamental properties of the DSA-EFIE system without relying on a particular discretization scheme, which can heavily impact properties such as low-frequency and dense-mesh breakdown and inadvertently introduces approximations by meshing or basis functions, scattering at a

(dielectric) sphere will be solved fully analytically. To this end, we rely on the complete set of vector spherical basis functions to expand all unknown quantities and operators, introduced for the PEC sphere [28], [42] in combination with the Surface-Volume-Surface EFIEs [29], [30]. Hence, we adopt the same notation and conventions as much as possible to facilitate the synergy with these publications.

A. Expansion of Surface Quantities in Spherical Basis Functions

The two unknowns on the boundary surface of the sphere in the problem at hand are $\mathbf{j}_s$ and $\mathbf{e}_0^t$. Considering a sphere with radius a about the origin, these quantities are naturally expanded in a Helmholtz decomposition through vector spherical harmonics [43] as:

$$\mathbf{j}_s = \sum_{n=0}^N \sum_{m=-n}^n \alpha_{nm}^{(1)} \mathbf{u}_{nm}^{(1)} + \alpha_{nm}^{(2)} \mathbf{u}_{nm}^{(2)} \quad (8)$$

$$\mathbf{e}_0^t = \sum_{n=0}^N \sum_{m=-n}^n \beta_{nm}^{(1)} \mathbf{u}_{nm}^{(1)} + \beta_{nm}^{(2)} \mathbf{u}_{nm}^{(2)}, \quad (9)$$

with $\mathbf{u}_{nm}^{(1)}$ the irrotational and $\mathbf{u}_{nm}^{(2)}$ the divergenceless component, defined as

$$\mathbf{u}_{nm}^{(1)} = \sqrt{d_{nm}} \nabla^t Y_n^m, \quad (10)$$

$$\mathbf{u}_{nm}^{(2)} = \sqrt{d_{nm}} \hat{\mathbf{r}} \times \nabla^t Y_n^m. \quad (11)$$

In these definitions, ∇^t is the surface gradient operator on a sphere with radius a

$$\nabla^t = \frac{1}{a} \frac{\partial}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{a \sin \theta} \frac{\partial}{\partial \varphi} \hat{\boldsymbol{\varphi}}, \quad (12)$$

Y_n^m are the scalar spherical harmonics and d_{nm} the normalization constants of the vector spherical harmonics:

$$Y_n^m(\theta, \varphi) = P_n^{|m|}(\cos \theta) e^{jm\varphi}, \quad (13)$$

$$d_{nm} = \frac{(n - |m|)!(2n + 1)}{4\pi(n + |m|)!n(n + 1)}, \quad (14)$$

with P_n^m the associated Legendre polynomials. Note the somewhat less conventional definition of the spherical harmonics in (13), introduced to ascertain the property $(Y_n^m)^* = Y_n^{-m}$, which will be used throughout the further derivations.

B. Expansion of the Green's Dyadic in Vector Spherical Harmonics

The Green's dyadic (3) can be expressed in terms of vector spherical harmonics complemented with a radial dependency [5]:

$$\begin{aligned} \bar{\bar{G}}_0(\mathbf{r}, \mathbf{r}') &= -jk_0 \sum_{n,m} d_{nm} \times \\ &\begin{cases} \mathbf{p}_{n,m}^{(1)}(\mathbf{r}) \mathbf{p}_{n,-m}^{(2)}(\mathbf{r}') + \mathbf{q}_{n,m}^{(1)}(\mathbf{r}) \mathbf{q}_{n,-m}^{(2)}(\mathbf{r}'), & r < r' \\ \mathbf{p}_{n,m}^{(2)}(\mathbf{r}) \mathbf{p}_{n,-m}^{(1)}(\mathbf{r}') + \mathbf{q}_{n,m}^{(2)}(\mathbf{r}) \mathbf{q}_{n,-m}^{(1)}(\mathbf{r}'), & r > r' \end{cases} \quad (15) \end{aligned}$$

where $r = ||\mathbf{r}'||$ & $r' = ||\mathbf{r}'||$, and with $\mathbf{p}^{(j)}$ and $\mathbf{q}^{(j)}$ defined as

$$\mathbf{p}_{nm}^{(j)} = -r z_n^{(j)}(k_0 r) \hat{\mathbf{r}} \times \nabla^t Y_n^m, \quad (16)$$

$$\mathbf{q}_{nm}^{(j)} = \frac{1}{k_0} \frac{\partial}{\partial k_0 r} \left[k_0 r z_n^{(j)}(k_0 r) \right]' \nabla^t Y_n^m + \frac{n(n+1)}{k_0 r} z_n^{(j)}(k_0 r) Y_n^m \hat{\mathbf{r}}, \quad (17)$$

where $[xf(x)]'$ denotes a derivative with respect to the full argument of f , i.e., $[xf(x)]' = f(x) + xf'(x)$, and z_n is the spherical Bessel function or the Hankel function of the second kind according to the following adopted convention:

$$z_n^{(1)}(x) = j_n(x) \quad z_n^{(2)}(x) = h_n^{(2)}(x). \quad (18)$$

Please remark that $\mathbf{p}$ and $\mathbf{q}$ are normalized in such a way that $k_0 \mathbf{p} = \nabla \times \mathbf{q}$ and $k_0 \mathbf{q} = \nabla \times \mathbf{p}$, where sub- and superscript are omitted for notational convenience.

C. Eigenfunctions of a PEC sphere

To express the DSA operator (7) in a spherical coordinate system, we require expressions for the magnetic eigenmodes $\mathbf{h}_\nu$ of a spherical PEC cavity with radius a . These eigenfunctions fall apart in two categories, transverse magnetic (TM) and transverse electric (TE) modes [43]. The former are given by

$$\mathbf{h}_{nms}^{\text{TM}} = \nabla \times [r j_n(k_{ns} r) Y_n^m(\theta, \varphi) \hat{\mathbf{r}}] = \tilde{\mathbf{p}}_{nms}^{(1)}, \quad (19)$$

where we have emphasized the similarity to $\mathbf{p}_{nm}^{(j)}$ introduced in Section III-B by $\tilde{\cdot}$, which constitutes a simple substitution of k_0 by $k_{ns} = x_{ns}/a$ with x_{ns} the roots of $[x j_n(x)]'$. Note that the summation index ν is here replaced by the written-out triple index $\{n, m, s\}$. Another characteristic required in (7) is the normalization constant whose square is given by [44]

$$\mathcal{N}_{nms}^2 = \frac{a^3}{2} \frac{1}{d_{nm}} \left[1 - \frac{n(n+1)}{x_{ns}^2} \right] j_n^2(x_{ns}). \quad (20)$$

The magnetic TE eigenfunctions can be expressed as

$$\mathbf{h}_{nms}^{\text{TE}} = \frac{1}{\kappa_{ns}} \nabla \times \nabla \times [r j_n(\kappa_{ns} r) Y_n^m(\theta, \varphi) \hat{\mathbf{r}}] = \tilde{\mathbf{q}}_{nms}^{(1)}, \quad (21)$$

where a similar resemblance as before has been observed to (17) by replacing the wavenumber k_0 by $\kappa_{ns} = y_{ns}/a$ with y_{ns} the roots of $j_n(x)$. The normalization constant of the second set is found to be

$$\mathcal{N}_{nms}^2 = \frac{a^3}{2} \frac{1}{d_{nm}} j_{n+1}^2(y_{ns}). \quad (22)$$

D. Galerkin testing of the system of equations

With the expansion of the core components fully defined, we find the exact solution of the DSA-EFIE system by applying method of moments (MoM) Galerkin testing to both (1) and (6). In the interest of enabling a thorough spectral analysis of the full system, we do not restrict ourselves to a particular

testing space at first instance and instead discretize the system for testing in the div-conforming Sobolev space H_{div}^l as

$$\langle \hat{\mathbf{n}} \times \mathbf{e}_0^t, \mathbf{u}_{\nu\mu}^{(i)} \rangle_{H_{\text{div}}^l} = \langle \hat{\mathbf{n}} \times \mathbf{e}_{\text{inc}}, \mathbf{u}_{\nu\mu}^{(i)} \rangle_{H_{\text{div}}^l} + \langle \mathcal{T} \circ \mathbf{j}_s, \mathbf{u}_{\nu\mu}^{(i)} \rangle_{H_{\text{div}}^l}, \quad (23)$$

$$\langle \mathbf{j}_s, \mathbf{u}_{\nu\mu}^{(i)} \rangle_{H_{\text{div}}^l} = \langle \mathcal{Y} \circ \mathbf{e}_0^t, \mathbf{u}_{\nu\mu}^{(i)} \rangle_{H_{\text{div}}^l}, \quad (24)$$

with $i = 1$ or 2 and the inner product defined as [28]

$$\langle \mathbf{a}, \mathbf{b} \rangle_{H_{\text{div}}^l} = \langle \mathbf{a}, \mathbf{b} \rangle_{H^l} + \langle \nabla \cdot \mathbf{a}, \nabla \cdot \mathbf{b} \rangle_{H^l}. \quad (25)$$

Computing the inner product over H^l for any order l can be cumbersome or outright tricky, in particular for negative orders. Fortunately, by expanding all unknowns and operators into (vector) spherical harmonics, we can rely on their properties to find simple expressions for the inner products in H_{div}^l and for their corresponding norm [28]. As will be shown in the remainder of this paper, all inner products are in some shape or form reducible to ones involving pairs of $\mathbf{u}_{nm}^{(i)}$. Hence, with the H^l norm for the vector spherical harmonics defined as

$$\langle \mathbf{u}_{nm}^{(i)}, \mathbf{u}_{\nu\mu}^{(j)} \rangle_{H^l} = (1 + n^2)^l \langle \mathbf{u}_{nm}^{(i)}, \mathbf{u}_{\nu\mu}^{(j)} \rangle_{L^2}, \quad (26)$$

and given their orthonormality in $L^2(S)$ (see their definition in (10)–(11)), the inner product in H_{div}^l is found to be

$$\langle \mathbf{u}_{nm}^{(i)}, \mathbf{u}_{\nu\mu}^{(j)} \rangle_{H_{\text{div}}^l(S)} = \delta_{ij} \psi_i^l \delta_{n\nu} \delta_{m\mu}, \quad (27)$$

with $\psi_1^l = (1 + n^2)^{l+1}$, $\psi_2^l = (1 + n^2)^l$. The difference between these factors for $\mathbf{u}_{nm}^{(1)}$ and $\mathbf{u}_{nm}^{(2)}$ is caused by the divergenceless nature of $\mathbf{u}^{(2)}$ (see (57)–(63) in [28] for more details). If we test in the traditional space $L^2(S)$, i.e., for $l = 0$, ψ_1^0 and ψ_2^0 are both equal to 1 and the inner product reduces to just the Kronecker delta functions.

E. Discretization of the EFIE

With the MoM testing and the generalized inner product fully defined, we can compute the matrix elements of (23) by substituting the expansions defined in Section III-A and Section III-C. Given the expansion of $\mathbf{e}_0^t$ (9), and the fact that $\mathbf{u}_{nm}^{(1),(2)} = \mp \hat{\mathbf{r}} \times \mathbf{u}_{nm}^{(2),(1)}$, the left-hand side evaluation of (23) always reduces to (27). Hence, a single term in the expansion of the tangential electric field will remain.

By introducing the expansion of the Green's dyadic (15) and $\mathbf{j}_s$ (8) in the last term, we find

$$\langle \mathcal{T} \circ \mathbf{u}_{nm}^{(j)}, \mathbf{u}_{\nu\mu}^{(i)} \rangle = -\omega \mu_0 k_0 \sum_{n', m'} d_{n' m'} \psi_{3-j}^l \psi_i^l \times \quad (28)$$

$$\left(\int_S \mathbf{p}_{n' m'}^{(2)*}(\mathbf{r}') \cdot \mathbf{u}_{nm}^{(j)}(\mathbf{r}') d\mathbf{r}' \int_S \hat{\mathbf{r}} \times \mathbf{p}_{n' m'}^{(1)}(\mathbf{r}) \cdot \left(\mathbf{u}_{\nu\mu}^{(i)}(\mathbf{r}) \right)^* d\mathbf{r} \right. \\ \left. + \int_S \mathbf{q}_{n' m'}^{(2)*}(\mathbf{r}') \cdot \mathbf{u}_{nm}^{(j)}(\mathbf{r}') d\mathbf{r}' \int_S \hat{\mathbf{r}} \times \mathbf{q}_{n' m'}^{(1)}(\mathbf{r}) \cdot \left(\mathbf{u}_{\nu\mu}^{(i)}(\mathbf{r}) \right)^* d\mathbf{r} \right).$$

Rewriting the integral over $\mathbf{r}'$ by shuffling the terms around, all integrals involved boil down to either of the four subsequent

integrals, whose calculations heavily rely on the orthogonality of the various spherical harmonics functions, to result in [28]:

$$\int_S \hat{\mathbf{r}} \times \mathbf{p}_{nm}^{(i)} \cdot (\mathbf{u}_{\nu\mu}^{(1)})^* d\mathbf{r} = \frac{az_n^{(i)}(k_0a)}{\sqrt{d_{nm}}} \delta_{n\nu} \delta_{m\mu}, \quad (29)$$

$$\int_S \hat{\mathbf{r}} \times \mathbf{p}_{nm}^{(i)} \cdot (\mathbf{u}_{\nu\mu}^{(2)})^* d\mathbf{r} = 0, \quad (30)$$

$$\int_S \hat{\mathbf{r}} \times \mathbf{q}_{nm}^{(i)} \cdot (\mathbf{u}_{\nu\mu}^{(1)})^* d\mathbf{r} = 0, \quad (31)$$

$$\int_S \hat{\mathbf{r}} \times \mathbf{q}_{nm}^{(i)} \cdot (\mathbf{u}_{\nu\mu}^{(2)})^* d\mathbf{r} = \frac{[k_0az_n^{(i)}(k_0a)]'}{k_0\sqrt{d_{nm}}} \delta_{n\nu} \delta_{m\mu}. \quad (32)$$

Applying these to (28), we find that only two of the four possible basis & test functions combinations do not vanish:

$$\langle \mathcal{T} \circ \mathbf{u}_{nm}^{(1)}, \mathbf{u}_{\nu\mu}^{(1)} \rangle = 0, \quad (33)$$

$$\langle \mathcal{T} \circ \mathbf{u}_{nm}^{(1)}, \mathbf{u}_{\nu\mu}^{(2)} \rangle = -\omega\mu_0 k_0 \sum_{n',m'} d_{n'm'} \psi_2^l \psi_2^l \times \quad (34)$$

$$\int_S \hat{\mathbf{r}} \times \mathbf{q}_{n'm'}^{(2)}(\mathbf{r}') \cdot (\mathbf{u}_{nm}^{(2)}(\mathbf{r}'))^* d\mathbf{r}' \int_S \hat{\mathbf{r}} \times \mathbf{q}_{n'm'}^{(1)}(\mathbf{r}) \cdot (\mathbf{u}_{\nu\mu}^{(2)}(\mathbf{r}))^* d\mathbf{r},$$

$$\langle \mathcal{T} \circ \mathbf{u}_{nm}^{(2)}, \mathbf{u}_{\nu\mu}^{(1)} \rangle = +\omega\mu_0 k_0 \sum_{n',m'} d_{n'm'} \psi_1^l \psi_1^l \times \quad (35)$$

$$\int_S \hat{\mathbf{r}} \times \mathbf{p}_{n'm'}^{(2)}(\mathbf{r}') \cdot (\mathbf{u}_{nm}^{(1)}(\mathbf{r}'))^* d\mathbf{r}' \int_S \hat{\mathbf{r}} \times \mathbf{p}_{n'm'}^{(1)}(\mathbf{r}) \cdot (\mathbf{u}_{\nu\mu}^{(1)}(\mathbf{r}))^* d\mathbf{r},$$

$$\langle \mathcal{T} \circ \mathbf{u}_{nm}^{(2)}, \mathbf{u}_{\nu\mu}^{(2)} \rangle = 0. \quad (36)$$

Fully evaluating the non-zero contributions, we find:

$$\langle \mathcal{T} \circ \mathbf{u}_{nm}^{(1)}, \mathbf{u}_{\nu\mu}^{(2)} \rangle = -\frac{\omega\mu_0}{k_0} \psi_2^l \psi_2^l \delta_{n\nu} \delta_{m\mu} \times \quad (37)$$

$$[k_0aj_n(k_0a)]' [k_0ah_n^{(2)}(k_0a)]',$$

$$\langle \mathcal{T} \circ \mathbf{u}_{nm}^{(2)}, \mathbf{u}_{\nu\mu}^{(1)} \rangle = \frac{\omega\mu_0}{k_0} \psi_1^l \psi_1^l \delta_{n\nu} \delta_{m\mu} \times \quad (38)$$

$$k_0aj_n(k_0a) k_0ah_n^{(2)}(k_0a).$$

With all terms explored, we introduce the expansions for the two unknown surface quantities (8) & (9) and substitute them in (23). Fully resolving the Kronecker delta functions, we find a one-to-one correspondence between every expansion coefficient β of $\mathbf{e}_0^t$ and its counterpart α from $\mathbf{j}_s$ as follows:

$$\beta_{nm}^{(1)} = \gamma_{nm}^{(2)} - \frac{\omega\mu_0}{k_0} \psi_2^l [k_0aj_n(k_0a)]' [k_0ah_n^{(2)}(k_0a)]' \alpha_{nm}^{(1)}$$

$$= \gamma_{nm}^{(2)} + \mathcal{Z}_n^{1,2} \alpha_{nm}^{(1)}, \quad (39)$$

$$\beta_{nm}^{(2)} = \gamma_{nm}^{(1)} - \frac{\omega\mu_0}{k_0} \psi_1^l k_0aj_n(k_0a) k_0ah_n^{(2)}(k_0a) \alpha_{nm}^{(2)}$$

$$= \gamma_{nm}^{(1)} + \mathcal{Z}_n^{2,1} \alpha_{nm}^{(2)}, \quad (40)$$

with

$$\gamma_{nm}^{(1)} = -\frac{\langle \hat{\mathbf{n}} \times \mathbf{e}_{\text{inc}}, \mathbf{u}_{nm}^{(1)} \rangle}{\psi_1^l}, \quad \gamma_{nm}^{(2)} = \frac{\langle \hat{\mathbf{n}} \times \mathbf{e}_{\text{inc}}, \mathbf{u}_{nm}^{(2)} \rangle}{\psi_2^l}. \quad (41)$$

These excitation terms depend on the precise form of the arbitrary incoming field, and are thus only computed when looking at particular scattering configurations. In matrix form, this discretized form is summarized as

$$\begin{bmatrix} \beta_{nm}^{(1)} \\ \beta_{nm}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_n^{1,2} & 0 \\ 0 & \mathcal{Z}_n^{2,1} \end{bmatrix} \begin{bmatrix} \alpha_{nm}^{(1)} \\ \alpha_{nm}^{(2)} \end{bmatrix} + \begin{bmatrix} \gamma_{nm}^{(2)} \\ \gamma_{nm}^{(1)} \end{bmatrix}. \quad (42)$$

Remark that for the specific geometry under study, i.e., with full spherical symmetry, the matrix system is completely diagonal and the EFIE matrix elements are independent of m .

F. Discretization of the DSA operator

In a similar approach, we take a closer look at the right-hand side of (24) first to determine the general, full form of the inner product:

$$\langle \mathcal{Y} \circ \mathbf{u}_{nm}^{(j)}, \mathbf{u}_{\nu\mu}^{(i)} \rangle = -\frac{k^2 - k_0^2}{j\omega\mu_0} \sum_{\nu'} \frac{k_{\nu'}^2 \psi_j^l \psi_i^l}{(k_{\nu'}^2 - k^2)(k_{\nu'}^2 - k_0^2) \mathcal{N}_{\nu'}^2} \times \quad (43)$$

$$\int_S \hat{\mathbf{n}} \times \mathbf{h}_{\nu'}^*(\mathbf{r}') \cdot \mathbf{u}_{nm}^{(j)}(\mathbf{r}') d\mathbf{r}' \int_S \hat{\mathbf{n}} \times \mathbf{h}_{\nu'}(\mathbf{r}) \cdot (\mathbf{u}_{nm}^{(i)}(\mathbf{r}))^* d\mathbf{r}.$$

With the two distinct sets of eigenmodes, i.e., TM (19) and TE (21), eight separate possible combinations can be made. However, given the orthogonality relations between $\mathbf{u}_{nm}^{(i)}$ and $\tilde{\mathbf{p}}_{nm}^{(1)}/\tilde{\mathbf{q}}_{nm}^{(1)}$, it is easily discerned that only two combination will result in non-zero contributions, viz., TM modes with $i = j = 1$ and TE modes with $i = j = 2$. Hence, through application of (29) and (20), the full expressions for the former becomes:

$$\langle \mathcal{Y} \circ \mathbf{u}_{nm}^{(1)}, \mathbf{u}_{\nu\mu}^{(1)} \rangle = -\frac{k^2 - k_0^2}{j\omega\mu_0} \psi_1^l \psi_1^l \delta_{n\nu} \delta_{m\mu} \times \quad (44)$$

$$a^2 \sum_{s'} \frac{k_{nms'}^2 j_n^2(x_{ns'})}{d_{nm} (k_{nms'}^2 - k^2) (k_{nms'}^2 - k_0^2) \mathcal{N}_{nms'}^2}$$

$$= -\frac{k^2 - k_0^2}{j\omega\mu_0} \psi_1^l \psi_1^l \delta_{n\nu} \delta_{m\mu} \times$$

$$\frac{2}{a} \sum_{s'} \frac{k_{nms'}^2}{(k_{nms'}^2 - k^2) (k_{nms'}^2 - k_0^2) \left[1 - \frac{n(n+1)}{x_{ns'}^2}\right]},$$

while the latter can be written out through introducing (32) and (22) as

$$\langle \mathcal{Y} \circ \mathbf{u}_{nm}^{(2)}, \mathbf{u}_{\nu\mu}^{(2)} \rangle = -\frac{k^2 - k_0^2}{j\omega\mu_0} \psi_2^l \psi_2^l \delta_{n\nu} \delta_{m\mu} \times \quad (45)$$

$$\sum_{s'} \frac{([\kappa_{nms'} a j_n(\kappa_{nms'} a)]')^2}{d_{nm} (\kappa_{nms'}^2 - k^2) (\kappa_{nms'}^2 - k_0^2) \mathcal{N}_{nms'}^2}$$

$$= -\frac{k^2 - k_0^2}{j\omega\mu_0} \psi_2^l \psi_2^l \delta_{n\nu} \delta_{m\mu} \times$$

$$\frac{2}{a} \sum_{s'} \frac{\kappa_{nms'}^2}{(\kappa_{nms'}^2 - k^2) (\kappa_{nms'}^2 - k_0^2)}$$

where we employed the fact that $[y_{ns'} j_n(y_{ns'})]' = y_{ns'} j_n'(y_{ns'}) = -y_{ns'} j_{n+1}(y_{ns'})$ on the grounds of $y_{ns'}$ being the zeros of $j_n(z)$ and the recurrence relation for the spherical Bessel derivative [44] simplifies the expression in the last step.

With the non-vanishing contribution completely written out, we turn back to (24) and follow the same procedure as performed for the EFIE in Section III-E to obtain an analogue

one-to-one relation between the expansion coefficients of $\mathbf{j}_s$ and $\mathbf{e}_0^t$:

$$\alpha_{nm}^{(1)} = \frac{\psi_1^l}{j a \omega \mu_0} \times \sum_{s'} \frac{-2k_{nms'}^2 (k^2 - k_0^2)}{(k_{nms'}^2 - k^2) (k_{nms'}^2 - k_0^2) \left[1 - \frac{n(n+1)}{x_{ns'}^2}\right]} \beta_{nm}^{(1)}, \quad (46)$$

$$\alpha_{nm}^{(2)} = \frac{\psi_2^l}{j a \omega \mu_0} \sum_{s'} \frac{-2\kappa_{nms'}^2 (k^2 - k_0^2)}{(\kappa_{nms'}^2 - k^2) (\kappa_{nms'}^2 - k_0^2)} \beta_{nm}^{(2)}, \quad (47)$$

which can be written concisely in matrix form as

$$\begin{bmatrix} \alpha_{nm}^{(1)} \\ \alpha_{nm}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_n^{1,1} & 0 \\ 0 & \mathcal{Y}_n^{2,2} \end{bmatrix} \begin{bmatrix} \beta_{nm}^{(1)} \\ \beta_{nm}^{(2)} \end{bmatrix}. \quad (48)$$

For the spherical geometry under study, the discretized DSA operator is characterized by a diagonal matrix with elements that do not depend on m , as similarly derived for the EFIE in (42). Since the DSA operator handles the interior region, which is presumed sourceless, no impact of the excitation comes into play for this discretized operator.

IV. SPECTRAL PROPERTIES OF THE DSA-EFIE

A. Closed-form DSA matrix elements

With the two equations governing the problem at hand fully solved for a spherical geometry, its analytical solution is obtained through determination of the α and/or β coefficients of the combined system (42) and (48). Unfortunately, the terms $\mathcal{Y}_n^{1,1}$ and $\mathcal{Y}_n^{2,2}$ are difficult to evaluate accurately as they require the roots x_{ns} and y_{ns} , respectively. Moreover, the sums in (46) and (47) converge very slowly (see Section VI for examples). Hence, at this point the DSA operator seems an impractical approach to construct an integral operator since an intractable number of terms would be required to compute it accurately.

Therefore, a closed form for these two sums would benefit the accuracy of the solution greatly and would prove the exactness of the DSA-EFIE. In this section, we propose such a closed sum by applying the generalized Fourier series concept [31], in particular an extension of the Fourier-Bessel series [44] to forms involving spherical Bessel functions. Generally, a square-integrable function f defined on the interval $[A, B]$ can be expanded as

$$f(x) = \sum_{s=0}^{\infty} c_s \phi_s(x), \quad (49)$$

with the set of orthogonal basis function ϕ_s themselves square-integrable on the same interval and ideally forming a complete set. The coefficients are found as

$$c_s = \frac{\langle f, \phi_s \rangle_w}{\langle \phi_s, \phi_s \rangle_w}, \quad (50)$$

with the inner product for the generalized Fourier series defined as $\langle f, g \rangle_w = \int_A^B f(x) g^*(x) w(x) dx$.

For the spherical Bessel functions we define the integration interval as $[A, B] = [0, a]$, the kernel $w(x) = x^2$ to represent the radial dependency of the differential volumetric element,

and take $\phi_s(x) = j_n\left(\frac{x_{ns}}{a}x\right)$ for (46) and $\phi_s(x) = j_n\left(\frac{y_{ns}}{a}x\right)$ for (47). Focusing first on the sum in (46), we rewrite it to explicitly show the differential nature of the $\mathcal{Y}$ -operator:

$$\begin{aligned} f_{\text{TM}}(a) &= -2 \sum_s \frac{k_{nms}^2 (k^2 - k_0^2)}{(k_{nms}^2 - k^2) (k_{nms}^2 - k_0^2) \left[1 - \frac{n(n+1)}{x_{ns}^2}\right]} \\ &= 2 \sum_s \left[\frac{k^2}{\left(k^2 - \left(\frac{x_{ns}}{a}\right)^2\right) \left[1 - \frac{n(n+1)}{x_{ns}^2}\right]} - \frac{k_0^2}{\left(k_0^2 - \left(\frac{x_{ns}}{a}\right)^2\right) \left[1 - \frac{n(n+1)}{x_{ns}^2}\right]} \right], \end{aligned} \quad (51)$$

where the sum has been split in a part solely depending on the original material parameters, while the second term is governed by the background material.

Inspired by the form of the DSA operator for the cuboid [26] and cylinder [45] expressed with entire domain basis functions, we propose the following function $f_{\text{TM}}(r)$:

$$f_{\text{TM}}(r) = -(ka)^2 \frac{j_n(kr)}{[ka j_n(ka)]'} + (k_0 a)^2 \frac{j_n(k_0 r)}{[k_0 a j_n(k_0 a)]'}. \quad (52)$$

Its generalized Fourier series evaluated at $r = a$ equals (51). We show this here for the first term only, as the derivation for the second term is of course completely analogous. Employing the following modified version of integrals 10.22.4 & 10.22.5 [44]:

$$\begin{aligned} \int_0^a r^2 j_n(br) j_n(cr) dr &= \quad (53) \\ a^2 \frac{b j_{n+1}(ba) j_n(ca) - c j_n(ba) j_{n+1}(ca)}{b^2 - c^2}, \\ \int_0^a r^2 j_n^2(br) dr &= \frac{a^3}{2} [j_n(ba)^2 - j_{n-1}(ba) j_{n+1}(ba)], \end{aligned} \quad (54)$$

the recurrence relations for the spherical Bessel function's derivatives, and $[x_{ns} j_n(x_{ns})]' = 0$, we obtain the following expression for the denominator of (50):

$$\begin{aligned} \langle \phi_s, \phi_s \rangle_{r^2} &= \frac{a^3}{2} [j_n(x_{ns})^2 - j_{n-1}(x_{ns}) j_{n+1}(x_{ns})] \quad (55) \\ &= \frac{a^3}{2} \left[1 - \frac{n(n+1)}{x_{ns}^2}\right] j_n^2(x_{ns}). \end{aligned}$$

Through (53) and the relation $x_{ns} j_{n+1}(x_{ns}) = (n+1) j_n(x_{ns})$, the expression for the numerator of (50) is found as

$$\begin{aligned} \left\langle \frac{j_n(kr)}{[ka j_n(ka)]'}, \phi_s \right\rangle_{r^2} &= \quad (56) \\ \frac{1}{[ka j_n(ka)]'} a \frac{ka j_{n+1}(ka) - (n+1) j_n(ka)}{k^2 - \left(\frac{x_{ns}}{a}\right)^2} j_n(x_{ns}). \end{aligned}$$

Relying once more on the recurrence relations for the spherical Bessel functions to simplify the numerator as $-[ka j_n(ka)]'$, the complete coefficient c_s for the generalized Fourier series of the first term in (52) is found to be

$$c_{s,\text{TM}} = 2 \sum_{s=0}^{\infty} \frac{k^2}{\left(k^2 - \left(\frac{x_{ns}}{a}\right)^2\right) \left[1 - \frac{n(n+1)}{x_{ns}^2}\right]} \frac{j_n\left(\frac{x_{ns}}{a}r\right)}{j_n(x_{ns})}, \quad (57)$$

which evaluates exactly to the first sum in (51) for $r = a$. By repeating the derivation for $k \rightarrow k_0$, we have thus proved that f_{TM} (52) does indeed equal the sum in (51) when evaluated in $r = a$.

Analogously, we split the TE sum:

$$f_{\text{TE}}(a) = -2 \sum_s \frac{\kappa_{nms}^2 (k^2 - k_0^2)}{(\kappa_{nms}^2 - k^2)(\kappa_{nms}^2 - k_0^2)} = 2 \sum_s \frac{y_{ns}^2}{a^2} \left[\frac{1}{k^2 - \left(\frac{y_{ns}}{a}\right)^2} - \frac{1}{k_0^2 - \left(\frac{y_{ns}}{a}\right)^2} \right]. \quad (58)$$

As the starting point to find a closed form, we leverage the function

$$g_{\text{TE}}(r) = a \frac{j_n(kr)}{j_n(ka)} - a \frac{j_n(k_0 r)}{j_n(k_0 a)}. \quad (59)$$

The integrals involved in computing the generalized Fourier series for its first term are

$$\begin{aligned} \langle \varphi_s, \varphi_s \rangle_{r^2} &= \frac{a^3}{2} [j_n(y_{ns})^2 - j_{n-1}(y_{ns})j_{n+1}(y_{ns})] \\ &= \frac{a^3}{2} j_{n+1}^2(y_{ns}), \end{aligned} \quad (60)$$

$$\begin{aligned} \langle a \frac{j_n(kr)}{j_n(ka)}, \varphi_s \rangle_{r^2} &= \frac{a^3}{j_n(ka)} \frac{k j_{n+1}(ka) j_n(y_{ns}) - \frac{y_{ns}}{a} j_n(ka) j_{n+1}(y_{ns})}{k^2 - \left(\frac{y_{ns}}{a}\right)^2} \\ &= -a^3 \left(\frac{y_{ns}}{a}\right) \frac{j_{n+1}(y_{ns})}{k^2 - \left(\frac{y_{ns}}{a}\right)^2}. \end{aligned} \quad (61)$$

g_{TE} can thus be expressed as

$$g_{\text{TE}}(r) = -2 \sum_s \left(\frac{y_{ns}}{a}\right) \left[\frac{1}{k^2 - \left(\frac{y_{ns}}{a}\right)^2} - \frac{1}{k_0^2 - \left(\frac{y_{ns}}{a}\right)^2} \right] \times \frac{j_n\left(\frac{y_{ns}}{a} r\right)}{j_{n+1}(y_{ns})}. \quad (62)$$

By taking the derivative of this expression, evaluating it at $r = a$ and simplifying $j'_n(y_{ns})$ as $-j_{n+1}(y_{ns})$, we find our closed-form expression for (58):

$$f_{\text{TE}}(a) = \frac{[ka j_n(ka)]'}{j_n(ka)} - \frac{[k_0 a j_n(k_0 a)]'}{j_n(k_0 a)}. \quad (63)$$

To summarize, the expressions for the two types of matrix elements in the discretized $\mathcal{Y}$ -operator (48) derived in Section III-F involving infinite sums, were found to have closed-form expressions, being the following, numerically much more accurate and easy-to-handle formulas:

$$\mathcal{Y}_n^{1,1} = -\frac{\psi_1^l}{j a \omega \mu_0} \left[\frac{(ka)^2 j_n(ka)}{[ka j_n(ka)]'} - \frac{(k_0 a)^2 j_n(k_0 a)}{[k_0 a j_n(k_0 a)]'} \right] \quad (64)$$

$$\mathcal{Y}_n^{2,2} = \frac{\psi_2^l}{j a \omega \mu_0} \left[\frac{[ka j_n(ka)]'}{j_n(ka)} - \frac{[k_0 a j_n(k_0 a)]'}{j_n(k_0 a)} \right] \quad (65)$$

B. Complete system matrix and eigenvalues

The two matrix equations (42) and (48) are easily combined into one by eliminating a set of unknowns, e.g., α , to obtain a complete system of equations:

$$\begin{bmatrix} \beta_{nm}^{(1)} \\ \beta_{nm}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_n^{1,2} \mathcal{Y}_n^{1,1} & 0 \\ 0 & \mathcal{Z}_n^{2,1} \mathcal{Y}_n^{2,2} \end{bmatrix} \begin{bmatrix} \beta_{nm}^{(1)} \\ \beta_{nm}^{(2)} \end{bmatrix} + \begin{bmatrix} \gamma_{nm}^{(2)} \\ \gamma_{nm}^{(1)} \end{bmatrix}. \quad (66)$$

Owing to the diagonal nature of all matrix elements involved, the explicit solution for the unknown coefficients is trivially found as

$$\beta_{nm}^{(1)} = \frac{\gamma_{nm}^{(2)}}{1 - \mathcal{Z}_n^{1,2} \mathcal{Y}_n^{1,1}} \quad \beta_{nm}^{(2)} = \frac{\gamma_{nm}^{(1)}}{1 - \mathcal{Z}_n^{2,1} \mathcal{Y}_n^{2,2}}, \quad (67)$$

which gives the solution to any scattering problem at a (dielectric) sphere for a given incoming field $\mathbf{e}_{\text{inc}}$.

Taking a step back, it is easily shown for the EFIE operator, through (1) with $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, (8)–(11), and (42) that

$$\mathcal{T} \circ (\nabla^t \mathbf{Y}_n^m) = \mathcal{Z}_n^{1,2} \hat{\mathbf{r}} \times \nabla^t \mathbf{Y}_n^m, \quad (68)$$

$$\mathcal{T} \circ (\hat{\mathbf{r}} \times \nabla^t \mathbf{Y}_n^m) = -\mathcal{Z}_n^{2,1} \nabla^t \mathbf{Y}_n^m. \quad (69)$$

Analogously for the DSA operator, (6), (8)–(11) and the matrix equation (48) show that the following relations hold

$$\mathcal{Y} \circ (\nabla^t \mathbf{Y}_n^m) = \mathcal{Y}_n^{1,1} \nabla^t \mathbf{Y}_n^m, \quad (70)$$

$$\mathcal{Y} \circ (\hat{\mathbf{r}} \times \nabla^t \mathbf{Y}_n^m) = \mathcal{Y}_n^{2,2} \hat{\mathbf{r}} \times \nabla^t \mathbf{Y}_n^m. \quad (71)$$

Consequently, after concatenation of (1) and (6), the total system equation

$$\hat{\mathbf{n}} \times \mathbf{e}_0^t = \hat{\mathbf{n}} \times \mathbf{e}_{\text{inc}} + \mathcal{T} \circ \mathcal{Y} \circ \mathbf{e}_0^t, \quad (72)$$

has eigenvalues

$$\Lambda_n^{(1)} = 1 - \mathcal{Z}_n^{1,2} \mathcal{Y}_n^{1,1}, \quad (73)$$

$$\Lambda_n^{(2)} = 1 - \mathcal{Z}_n^{2,1} \mathcal{Y}_n^{2,2}, \quad (74)$$

which shows that the denominators of the system's solutions (67) coincide with the eigenvalues of the DSA-EFIE.

C. Oversampling breakdown

As the system's eigenvalues contain the product of the two individual operator's eigenvalues, the asymptotic behavior for $n \rightarrow \infty$ is determined by their respective dominant term for large n . For the EFIE components, the large-order expansions of the Bessel functions — see (A.1) and (A.2) in Appendix A — readily leads to

$$\mathcal{Z}_n^{1,2} \sim n \psi_2^l \frac{j}{2k_0 a}, \quad \mathcal{Z}_n^{2,1} \sim \frac{\psi_1^l}{n} \frac{j k_0 a}{2}. \quad (75)$$

For the DSA eigenvalues, the ratios of two spherical Bessel functions require a more careful approach as outlined in Appendix A. For the first set of eigenvalues, the first-order approximation suffices and results in

$$\begin{aligned} \mathcal{Y}_n^{1,1} &\sim -\frac{\psi_1^l}{j a \omega \mu_0} \left[\frac{(ka)^2}{ka \frac{n}{ka} + 1} - \frac{(k_0 a)^2}{k_0 a \frac{n}{k_0 a} + 1} \right], \\ &\sim -\frac{\psi_1^l}{n} a \frac{k^2 - k_0^2}{j \omega \mu_0}. \end{aligned} \quad (76)$$

In determining the asymptotic behavior for large n for the second set of DSA eigenvalues, the second order approximation (A.15) is needed and leads to

$$\begin{aligned} \mathcal{Y}_n^{2,2} &\sim \frac{\psi_2^l}{j a \omega \mu_0} \left[ka \left(\frac{n}{ka} - \frac{ka}{2n} \right) - k_0 a \left(\frac{n}{k_0 a} - \frac{k_0 a}{2n} \right) \right], \\ &\sim \frac{\psi_2^l}{n} a \frac{k^2 - k_0^2}{j \omega \mu_0}. \end{aligned} \quad (77)$$

By combining these results, the eigenvalues of the DSA-EFIE system thus behave asymptotically, i.e., for large order n , as

$$\Lambda_n^{(1)} \sim 1 + \sigma \psi_2^l \psi_1^l \sim \begin{cases} 1 - \sigma(1 + n^2)^{2l+1} & l \neq 0, \\ 1 + \sigma & l = 0, \end{cases} \quad (78)$$

$$\Lambda_n^{(2)} \sim 1 - \varsigma \frac{\psi_1^l \psi_2^l}{n^2} \sim \begin{cases} 1 - \varsigma(1 + n^2)^{2l} & l \neq 0, \\ \mathcal{O}(1) & l = 0, \end{cases} \quad (79)$$

with the constants $\sigma = (k^2 - k_0^2)/(2j\omega k_0 \mu_0)$ and $\varsigma = k_0 a^2 (k^2 - k_0^2)/(4j\omega \mu_0)$ independent of n . From this result, we can draw some important conclusions. Although the DSA-EFIE has been successfully applied to a variety of EM modeling problems [19], [22], [23], [25], [26], [41], [46], it shows for the first time in literature that the formulation itself is inherently free from oversampling breakdown. When the standard $L^2(S)$ space is chosen ($l = 0$), $\psi_1^0 = \psi_2^0 = 1$; hence, the condition number remains bounded since $\Lambda^{(1)}$ evolves to $1 + \sigma$ while $\Lambda^{(2)}$ accumulates at one. Testing in the EFIE's required $l = -\frac{1}{2}$ space [28], makes it so that both $\Lambda^{(1)}$ and $\Lambda^{(2)}$ accumulate at one, leading to a bounded condition number as well. Hence the inclusion of the DSA operator does not affect the dense-mesh breakdown of the EFIE in a negative way; quite the contrary, it opens up the possibility to test in a standard function space instead of a more challenging Sobolev space.

D. Low-frequency breakdown

For decreasing frequencies, the argument of the Bessel and Hankel functions in the expressions for the eigenvalues tends towards zero. Hence, by introducing the small-argument form of these functions (see (B.1) and (B.2) in Appendix B), and once more taking precautions for the $\mathcal{Y}$ -values by including the second-order approximation when required, the limiting behavior is found to be

$$\begin{aligned} \mathcal{Z}_n^{1,2} &\sim \omega^{-1}, & \mathcal{Y}_n^{1,1} &\sim \omega^1, \\ \mathcal{Z}_n^{2,1} &\sim \omega^1, & \mathcal{Y}_n^{2,2} &\sim \omega^1. \end{aligned} \quad (80)$$

This leads to an overall low-frequency tendency of

$$\Lambda_n^{(1)} \sim 1 - \omega^0, \quad (81)$$

$$\Lambda_n^{(2)} \sim 1 - \omega^2, \quad (82)$$

which solves the typical low-frequency behavior of the EFIE as both set of eigenvalues now accumulate at a finite value. Note that the testing space has no effect on these results as the ψ_i^l factors are frequency-independent. Nevertheless, methods designed to deal with low-frequency breakdown, such as the augmented EFIE [47] or preconditioners [48] could still be invoked to improve the condition number and optimize performance in iterative solvers.

E. Internal resonances

The eigenvalues $\Lambda_n^{(1)}$ and $\Lambda_n^{(2)}$ become zero or infinite at certain, discrete frequencies caused by either their $\mathcal{Z}$ or $\mathcal{Y}$ component, respectively, or when $\mathcal{Z}_n^{1,2} \mathcal{Y}_n^{1,1} = 1$ or $\mathcal{Z}_n^{2,1} \mathcal{Y}_n^{2,2} = 1$. This phenomenon is commonly referred to as internal resonance breakdown. The EFIE's eigenvalues (42)

never become infinite but, given the form of the DSA's eigenvalues (48), an internal resonance does occur when $k_0 a = x_{ns}$ for $\mathcal{Y}^{1,1}$ or when $k_0 a = y_{ns}$ for $\mathcal{Y}^{2,2}$. Hence, when the frequency coincides with one of the eigenmodes of a PEC cavity of size $\mathcal{V}$, the typical EFIE internal resonances remain. Moreover, for the DSA values, the denominator in its general form (7) clearly reveals potential issues as well when $ka = k_{nms}$. This condition does not pose problems if the material is lossy since k_{nms} is real for all eigenmodes. Lastly, when the transcendental equation (73) or (74) has a root, an additional internal resonance takes place. Given the complex nature of the relation, no closed form for these resonances has been found. In conclusion, this analysis rigorously shows that the DSA-EFIE does not resolve the internal resonance issue plaguing single-source formulations.

V. EQUIVALENCE TO THE MIE SERIES

To prove that the DSA-EFIE tackles the scattering problem for dielectric objects exactly, we will now establish its mathematical equivalence to the Mie series solution for scattering by a plane wave at any frequency and for any dielectric sphere when taking the norm in the conventional $L^2(S)$. Given the symmetry of the geometry and the fact that the plane wave is a fundamental solution of the wave equation, this proof for one single incident plane wave can be generalized to any arbitrary excitation. Consider thus an x -polarized wave propagating along the positive z -axis, i.e., $\mathbf{e}_{\text{inc}} = \hat{\mathbf{x}}e^{-jk_0 z}$. The Mie series for this configuration can be found in various reference texts, for example [4], [43], where we will adopt the form as presented in Section 11.3 of the latter.

The total electric field $\mathbf{e}$ in- and outside of the sphere can be expressed in terms of the Debye potentials v and w as $\mathbf{e} = \nabla \times \nabla \times (rv\hat{\mathbf{r}}) - j\omega\mu_0 \nabla \times (rw\hat{\mathbf{r}})$, whose tangential components on the surface of the sphere become

$$\mathbf{e}_0^t = \frac{\partial}{\partial r} (r\nabla^t v) \Big|_{r=a} + j\omega\mu_0 a \hat{\mathbf{r}} \times \nabla^t w. \quad (83)$$

The two potentials [43] can be expressed in terms of the scalar spherical harmonics Y_n^m (13), yielding the following expressions on S

$$v = \frac{-1}{j\epsilon_r k_0} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} (-j)^n v_n \frac{(Y_n^1 + Y_n^{-1})}{2} j_n(kr), \quad (84)$$

$$w = \frac{1}{\omega\mu_0} \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} (-j)^n w_n \frac{(Y_n^1 - Y_n^{-1})}{2} j_n(kr), \quad (85)$$

with the expansion coefficients v_n and w_n given by

$$v_n = \frac{j_n(k_0 a) \left[k_0 a h_n^{(2)}(k_0 a) \right]' - h_n^{(2)}(k_0 a) [k_0 a j_n(k_0 a)]'}{j_n(ka) \left[k_0 a h_n^{(2)}(k_0 a) \right]' - \frac{h_n^{(2)}(k_0 a)}{\epsilon_r} [ka j_n(ka)]'}, \quad (86)$$

$$w_n = \frac{j_n(k_0 a) \left[k_0 a h_n^{(2)}(k_0 a) \right]' - h_n^{(2)}(k_0 a) [k_0 a j_n(k_0 a)]'}{j_n(ka) \left[k_0 a h_n^{(2)}(k_0 a) \right]' - h_n^{(2)}(k_0 a) [ka j_n(ka)]'}. \quad (87)$$

In order to obtain the equivalent expression for the DSA-EFIE solution (9), we need the expansion coefficients β_{nm} (67) and thus also the excitation terms γ_{nm} (41) for the incident plane wave. The expansion of a plane wave into spherical harmonics is well-established (see e.g., Section 10.6 in [43]) and careful examination of this representation shows that it is noted down compactly in terms of the functions $\mathbf{p}^{(1)}$ (16) and $\mathbf{q}^{(1)}$ (17) as

$$\mathbf{e}_{\text{inc}} = \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \frac{j}{2} \left[\mathbf{q}_{n,1}^{(1)} + \mathbf{q}_{n,-1}^{(1)} - \mathbf{p}_{n,1}^{(1)} + \mathbf{p}_{n,-1}^{(1)} \right]. \quad (88)$$

Hence, we can rely once again on the orthogonality relations of the spherical harmonics functions (29)–(32) in computing γ_{nm} , which yields

$$\gamma_{n,m}^{(1)} = m(-j)^n \frac{2n+1}{n(n+1)} \frac{j}{2} \frac{a j_n(k_0 a)}{\sqrt{d_{n,1}}} \delta_{m,|1|}, \quad (89)$$

$$\gamma_{n,m}^{(2)} = (-j)^n \frac{2n+1}{n(n+1)} \frac{j}{2} \frac{1}{k_0 \sqrt{d_{n,1}}} [k_0 a j_n(k_0 a)]' \delta_{m,|1|}. \quad (90)$$

The Kronecker deltas reduce the double summation in the expansion of the tangential electric field (9) to

$$\begin{aligned} \mathbf{e}_0^t &= \sum_{n=1}^N \frac{2\sqrt{d_{n,1}} \gamma_{n,1}^{(2)}}{1 - \mathcal{Z}_n^{1,2} \mathcal{Y}_n^{1,1}} \nabla^t (Y_n^1 + Y_n^{-1}) \\ &\quad + \frac{2\sqrt{d_{n,1}} \gamma_{n,1}^{(1)}}{1 - \mathcal{Z}_n^{2,1} \mathcal{Y}_n^{2,2}} \hat{\mathbf{r}} \times \nabla^t (Y_n^1 - Y_n^{-1}). \end{aligned} \quad (91)$$

A term-by-term comparison between the Mie series (83)–(85) and the DSA-EFIE expression (91) reveals that the following two relations need to be proven to establish complete equivalence:

$$v_n = \frac{\epsilon_r [k_0 a j_n(k_0 a)]'}{[k_0 a j_n(k_0 a)]'} \frac{1}{1 - \mathcal{Z}_n^{1,2} \mathcal{Y}_n^{1,1}}, \quad (92)$$

$$w_n = \frac{j_n(k_0 a)}{j_n(k_0 a)} \frac{1}{1 - \mathcal{Z}_n^{2,1} \mathcal{Y}_n^{2,2}}. \quad (93)$$

To achieve this, we rely repeatedly on the Wronskian for spherical Bessel functions [44]:

$$j_n(k_0 a) \left[k_0 a h_n^{(2)}(k_0 a) \right]' - h_n^{(2)}(k_0 a) [k_0 a j_n(k_0 a)]' = \frac{-j}{k_0 a}. \quad (94)$$

Multiplying the denominator on the right-hand side of (92) with $-j/k_0 a$, leveraging the Wronskian (94), and simplifying, we find it to be:

$$\begin{aligned} &\frac{-j}{k_0 a} [k_0 a j_n(k_0 a)]' + [k_0 a h_n^{(2)}(k_0 a)]' \epsilon_r j_n(k_0 a) [k_0 a j_n(k_0 a)]' \\ &\quad - [k_0 a h_n^{(2)}(k_0 a)]' j_n(k_0 a) [k_0 a j_n(k_0 a)]' \end{aligned} \quad (95)$$

$$\begin{aligned} &= \left(j_n(k_0 a) [k_0 a h_n^{(2)}(k_0 a)]' - \frac{h_n^{(2)}(k_0 a)}{\epsilon_r} [k_0 a j_n(k_0 a)]' \right) \times \\ &\quad \epsilon_r [k_0 a j_n(k_0 a)]'. \end{aligned} \quad (96)$$

Multiplying the numerator of (92) with the expanded form of the Wronskian as well, it is clear that relation (92) indeed holds true.

To prove (93), we follow a very similar method, simplifying the denominator first through the multiplication of the Wronskian:

$$\begin{aligned} &\frac{-j}{k_0 a} j_n(k_0 a) - j_n(k_0 a) h_n^{(2)}(k_0 a) [k_0 a j_n(k_0 a)]' \\ &\quad + j_n(k_0 a) h_n^{(2)}(k_0 a) [k_0 a j_n(k_0 a)]' \end{aligned} \quad (97)$$

$$\begin{aligned} &= \left(j_n(k_0 a) [k_0 a h_n^{(2)}(k_0 a)]' - h_n^{(2)}(k_0 a) [k_0 a j_n(k_0 a)]' \right) \times \\ &\quad j_n(k_0 a). \end{aligned} \quad (98)$$

Multiplying the numerator of (93) with (94) as the final step, it becomes apparent that this relation holds exactly as well.

Given the mathematical equivalence of the Mie reference solution and the DSA-EFIE expression for the tangential electric field on the entire surface of the sphere, the field distribution of the entire space, both inside and outside the sphere, is established to be the same as well through application of the equivalence theorem [4], for any dielectric material and frequency.

VI. NUMERICAL EXPERIMENTS

In this section, we present some numerical experiments that showcase the theoretical results of Sections III–IV. First, we demonstrate the advantage of the closed sums derived in Section IV-A through an investigation of the convergence rate of the pertinent series. Additionally, we exhibit the confirmed breakdowns (or lack thereof) for a particular example. These results are not included to serve as a validation of the mathematical derivations, which was taken care of in Section V instead, but as an aid in the interpretation of the main conclusions presented in the above derivations.

In Section IV-A, closed expressions (64) and (65) for the infinite sums in $\mathcal{Y}_n^{1,1}$ and $\mathcal{Y}_n^{2,2}$, respectively, were derived. To demonstrate the efficacy of these expressions, we have provided the relative error of the summation form of both DSA elements, i.e., (46) and (47), for $n = 4$ compared to the closed form for $k_0 a = 4\pi$ for a sphere with radius $a = 1$ m. The convergence of the sums for a low-contrast dielectric with $\epsilon_r = 10$, a high-contrast material with $\epsilon_r = 100$ and copper ($\sigma = 5.8 \cdot 10^7$ S/m), is shown in Figure 2. It is immediately clear that the analytical expressions provide a huge accuracy and efficiency advantage given the (eventual) slow s^{-1} convergence of the sums, which is further aggravated for the high-contrast material and most definitely for copper where a minimum of 250 and $4 \cdot 10^5$ eigenmodes, respectively, are required to achieve one digit of significance.

Next, we illustrate the spectral properties of the DSA-EFIE formulation by looking at the limits derived in Sections IV-C–IV-E for $l = 0$, i.e., in the standard $L^2(S)$ -space. For the same sphere with radius 1 m and for a dielectric with $\epsilon_r = 3$, the two sets of eigenvalues are plotted in Figure 3a for $k_0 a = 4\pi$ with an increasing order. After transient behavior up to $n = 19$, both system eigenvalues start to converge to a constant value, hence confirming the lack of inherent breakdown of the combined DSA-EFIE system as derived in (78)–(79). In Figure 3b, the order of the eigenvalues is fixed at $n = 4$ and they are graphed out from 10 kHz to 1 GHz. The dominant behavior

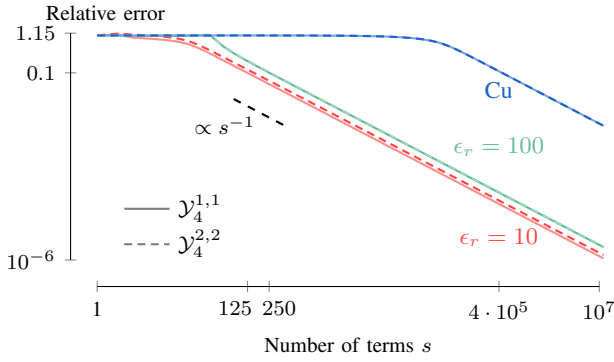


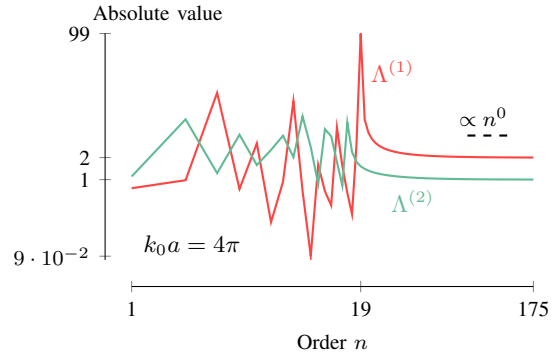
Fig. 2. Relative error of $Y_n^{1,1}$ (46) and $Y_n^{2,2}$ (47) compared to their closed expressions (64) and (65), respectively, for a low-contrast material with $\epsilon_r = 10$ (red), a high-contrast dielectric with $\epsilon_r = 100$ (green) and copper with $\sigma = 5.8 \cdot 10^7$ S/m (blue).

derived in (81)-(82) for both sets is confirmed as both graphs level out as the frequency decreases, showing the lack of low-frequency breakdown. Lastly, in Figure 3c, the eigenvalues are plotted for $n = 0$ for the frequency range encompassing the first five internal resonances. The four peaks to infinity correspond to material resonances in $Y^{1,1}$ and $Y^{2,2}$ since we are looking at a lossless material — hence the factor $\sqrt{\epsilon_r}$ in the denominator. The resonance to zero corresponds with $Z^{1,2}Y_0^{1,1} = 1$ and coincides with the zero $x_{01}c$ of both operators involved because of the particular form of spherical Bessel functions for $n = 0$; for other orders, these zeros do not have closed expressions. Remarkably, a joint resonance of $Z^{2,1}$ and $Y^{2,2}$ at $y_{01}c$ has been eliminated in the concatenated system.

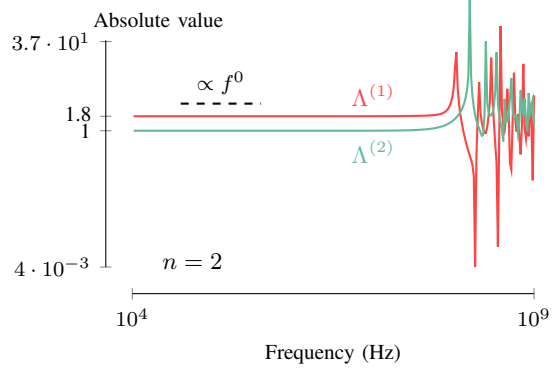
VII. CONCLUSION

This work introduces a complete and exact solution of the DSA-EFIE formulation for scattering by a nonmagnetic dielectric sphere using a Galerkin MoM, to study this BIE formulation's spectral properties. By expanding the unknown field quantities on the sphere's surface, along with the EFIE and DSA operator, into vector spherical harmonics, their orthogonality has led to a fully analytical computation of the relevant MoM integrals. This choice of basis and test functions allowed for a thorough investigation of the overall system's spectral performance by introducing testing in different Sobolev functional spaces. Additionally, the discretized DSA elements have been transformed from infinite series into simple, closed analytical expressions through the utilization of the newly adapted generalized Fourier series for spherical Bessel functions, ultimately leading to convenient formulas for the operator's eigenvalues, facilitating their interpretation, the proof of the equivalence to the Mie series, and the consequent spectral analysis tremendously.

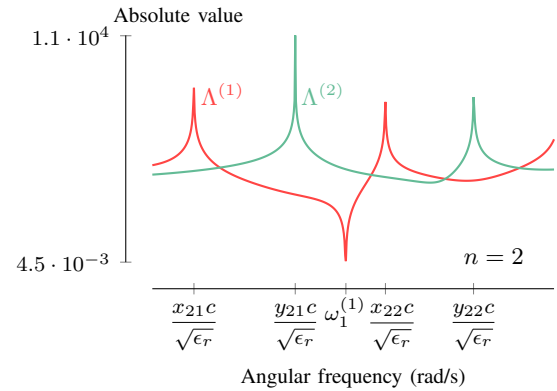
This has resulted in five insights into the DSA-EFIE. First, the method was definitively proven to rigorously solve Maxwell's equations at any frequency for any dielectric by showing the solution's equivalence to the classical Mie series. Second, the closed sums of the discretized DSA elements enable the seamless inclusion of any dielectric material, including good conductors, which are typically challenging to



(a)



(b)



(c)

Fig. 3. Eigenvalue behavior of the discretized EFIE-DSA operator elements for $\epsilon_r = 3$; dense mesh behavior for $k_0 a = 4\pi$ (a), low-frequency limit for $n = 4$ (b), and internal resonances for $n = 0$ (c).

handle accurately. Third, the spectral analysis has precisely determined the breakdown behavior of the pertinent integral operators as well as that of the complete system for the first time, backed by the mathematical framework. This showed that the condition number of the DSA-EFIE will remain bounded with the order n , for both the space of square integrable functions and for the Sobolev space $H_{\text{div}}^{-1/2}$ as the two sets of eigenmodes have finite accumulation points. Fourth, the DSA-EFIE is spared from low-frequency breakdown in all testing spaces. Finally, it is shown to suffer from internal resonances, a typical issue with single-source integral equation, which is not resolved by the introduction of the DSA

operator, although it changes the exact resonance frequencies.

Although the DSA-EFIE has proven its worth in plenty of examples and applications, these results on its fundamental, spectral properties now show its merits and shortcomings rigorously. Additionally, they provide an analysis avenue to assess alternative solutions that extend the modeling capabilities or present improved performance. Hence, future work will focus on amending the possibilities outlined above and extending the appositeness of the DSA operator.

APPENDIX A

BESSEL FUNCTION EXPANSION FOR LARGE ORDERS

The (non-spherical) Bessel functions have the well-known first-order expansions for large orders [44]:

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu, \quad (\text{A.1})$$

$$H_\nu^{(2)}(z) \sim j\sqrt{\frac{2}{\pi\nu}} \left(\frac{ez}{2\nu}\right)^{-\nu}. \quad (\text{A.2})$$

However, in some asymptotic eigenvalue expression the second order is required as cancellation between different terms occurs. To find a more general, non-standard expression, we start from Poisson's integral representation of the Bessel function in the non-trigonometric form [44]:

$$J_\nu(z) = \frac{z^\nu}{2^\nu \sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(zt) dt, \quad (\text{A.3})$$

which is valid as long as $\nu > -\frac{1}{2}$. Transforming it through $-u^2 = \ln(1-t^2)$, this becomes:

$$J_\nu(z) = \frac{z^\nu}{2^\nu \sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{-\infty}^{\infty} \frac{\cos(z\sqrt{1-e^{-u^2}})|u|e^{-u^2/2}}{\sqrt{1-e^{-u^2}}} e^{-\nu u^2} du. \quad (\text{A.4})$$

Grouping all terms of the integrand not dependent on ν into the auxiliary function $f(u)$

$$f(u) = \frac{\cos(z\sqrt{1-e^{-u^2}})|u|e^{-u^2/2}}{\sqrt{1-e^{-u^2}}}, \quad (\text{A.5})$$

the integral is now formulated as a function times a Gaussian kernel

$$J_\nu(z) = \frac{z^\nu}{2^\nu \sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{-\infty}^{\infty} f(u) e^{-\nu u^2} du. \quad (\text{A.6})$$

As we take the limit for $\nu \rightarrow \infty$, the Gaussian kernel will approach a Dirac distribution, hence allowing for an approximation of $f(u)$ by its Taylor series before integrating. In the end, we can approximate our Bessel function by expanding the gamma function around $\nu = \infty$ and $f(u)$ around the origin up to the number of terms we require.

The expansion of the fraction in front of the integral is based on the asymptotic expansion of the gamma function for large arguments [44]:

$$\Gamma(z) = \left(\frac{z}{e}\right)^z \frac{\sqrt{2\pi}}{\sqrt{z}} \sum_{k=0}^{\infty} \frac{g_k}{z^k}, \quad (\text{A.7})$$

with the coefficients g_k based on the partial Bell polynomials. However, we require the expansion for the reciprocal of Γ . In [49], it is shown that this expansion can be constructed based on (A.7) through a sign change of the coefficients as

$$\frac{1}{\Gamma(z)} = \left(\frac{z}{e}\right)^{-z} \frac{\sqrt{z}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{g_k}{z^k}. \quad (\text{A.8})$$

To finally obtain the large argument expansion for $\Gamma(\nu + \frac{1}{2})$, we rely on the Taylor series of $h(z) = (1+z/2)^{-1/z}$ around $z = 0$ to find

$$\begin{aligned} \frac{1}{\Gamma(\nu + \frac{1}{2})} &= \left(\frac{\nu}{e}\right)^{-\nu} \frac{\sqrt{e}}{\sqrt{2\pi}} \left(1 + \frac{1}{2\nu}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k g_k}{(\nu + \frac{1}{2})^k} \\ &= \left(\frac{\nu}{e}\right)^{-\nu} \frac{\sqrt{e}}{\sqrt{2\pi}} h\left(\frac{1}{\nu}\right) \sum_{k=0}^{\infty} \frac{(-1)^k g_k}{(\nu + \frac{1}{2})^k} \\ &\sim \left(\frac{\nu}{e}\right)^{-\nu} \frac{1}{\sqrt{2\pi}} \left(1 + \frac{1}{8\nu} + \mathcal{O}(\nu^{-2})\right) \sum_{k=0}^{\infty} \frac{(-1)^k g_k}{\nu^k} \\ &\sim \left(\frac{\nu}{e}\right)^{-\nu} \frac{1}{\sqrt{2\pi}} \left[1 + \frac{1}{24\nu} + \mathcal{O}(\nu^{-2})\right]. \end{aligned} \quad (\text{A.9})$$

The Taylor series of $f(u)$ around $u = 0$ is found to be

$$f(u) = 1 - \frac{2z^2 + 1}{4} u^2 + \mathcal{O}(u^4), \quad (\text{A.10})$$

which, after integrating with the Gaussian kernel, leads to

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(u) e^{-\nu u^2} du = \frac{1}{\nu^{1/2}} - \frac{2z^2 + 1}{8} \frac{1}{\nu^{3/2}} + \mathcal{O}(\nu^{-5/2}). \quad (\text{A.11})$$

Combining this integral with (A.9), the following asymptotic expression is eventually obtained for $J_\nu(z)$:

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi}} \left(\frac{ez}{2\nu}\right)^\nu \left[\nu^{-1/2} - \frac{3z^2 + 1}{12} \nu^{-3/2} + \mathcal{O}(\nu^{-5/2}) \right]. \quad (\text{A.12})$$

In Section IV-C, we actually require the expansion for $j'_n(z)/j_n(z)$, which we can derive from (A.12) by starting from $\ln(j_n(z))$. For the first-order approximation, this becomes:

$$\begin{aligned} \ln(j_n(z)) &= \ln\left(\sqrt{\frac{\pi}{2z}} J_{n+1/2}(z)\right) \\ &\sim \ln\left(\frac{1}{2} \left(\frac{e}{2}\right)^{n+1/2} (n+1/2)^{-n-1}\right) + n \ln(z) \end{aligned} \quad (\text{A.13})$$

and as such, it is shown that

$$[\ln(j_n(z))]' = \frac{j'_n(z)}{j_n(z)} \sim \frac{n}{z} \quad (\text{A.14})$$

In case we require the second-order term as well, we analogously find:

$$\frac{j'_n(z)}{j_n(z)} \sim \frac{n}{z} - \frac{6z}{12n - 3z^2 + 5} \sim \frac{n}{z} - \frac{z}{2n} \quad (\text{A.15})$$

APPENDIX B

BESSEL FUNCTION EXPANSION FOR SMALL ARGUMENTS

The expansion for small arguments centers around the following limiting form for small arguments of the spherical Bessel function [44]:

$$j_n(z) = \frac{z^n}{(2n+1)!!} - \frac{z^{n+2}}{2(2n+3)!!} + \mathcal{O}(z^{n+4}) \quad (\text{B.1})$$

$$h_n^{(2)}(z) = \frac{(2n+1)!!}{z^{n+1}} + \mathcal{O}(z^{-n+1}) \quad (\text{B.2})$$

with the double factorial defined as $n!! = n(n-2)(n-4) \cdots 2$ or $n!! = n(n-2)(n-4) \cdots 1$ depending on the parity of n .

In Section IV-D the expansion for $j'_n(z)/j_n(z)$ is required, which we can derive from (B.1) taken the derivative of $\ln(j_n(z))$. For the first-order approximation, this becomes

$$\begin{aligned} \ln(j_n(z)) &\sim \ln\left(\frac{z^n}{(2n+1)!!}\right) \\ \Rightarrow [\ln(j_n(z))]' &= \frac{j'_n(z)}{j_n(z)} \sim \frac{n}{z}, \end{aligned} \quad (\text{B.3})$$

while for the second-order behavior, one finds that

$$\frac{j'_n(z)}{j_n(z)} \sim \frac{n}{z} - \frac{z}{2n+3}. \quad (\text{B.4})$$

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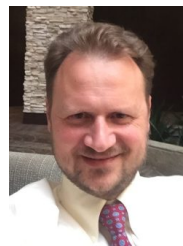


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